ISRAEL JOURNAL OF MATHEMATICS 113 (1999), 163–204

♣ DOES NOT IMPLY THE EXISTENCE OF A SUSLIN TREE*

BY

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ABSTRACT

We prove that ♣ does not imply the existence of a Suslin tree, so answering a question of I. Juhász.

1. Introduction

In his paper [Ost], A. J. Ostaszewski introduced the combinatorial principle ♣. The principle is a weaker simple relative of $\diamondsuit$ and has found many applications in set-theoretic topology; see [KuVa].

* The authors thank the Israel Academy of Sciences and Humanities for partial support through the Basic Research Foundation Grant number 0327398. Mirna Džamonja would in addition like to thank The Hebrew University of Jerusalem and The Lady Davis Foundation for the Forchheimer Postdoctoral Fellowship during the academic year 1994/95, and Rutgers University for their hospitality during a visit in November 1995, when part of the research for this paper was conducted. Some of the research was also conducted while Mirna Džamonja was a visiting assistant professor at the University of Wisconsin-Madison. This publication is denoted [DjSh 604] in the list of publications of Saharon Shelah.

Received December 25, 1996

Definition 1.1: $\clubsuit$ means that there is a sequence $\langle A_\delta \colon \delta \text{ limit } < \omega_1\rangle$ such that

(i) each A_δ is an unbounded subset of δ, and

(ii) for every $A \in [\omega_1]^{\aleph_1}$, there is δ such that $A_\delta \subseteq A$. (Equivalently: there are stationarily many such δ.) ∎

It is clear that $\diamondsuit \implies \clubsuit$, and it was already noted in [Ost] that $\clubsuit + CH$ implies $\diamondsuit$ (as explained in [Ost], the argument is due to K. Devlin (in [Sh 98] also Burgess is credited)). For a while, it was open if $\clubsuit$ and $\diamondsuit$ were actually equivalent, but this was settled by S. Shelah in [Sh 98], where a model of $\clubsuit$ is constructed in which CH does not hold. The proof starts with $V = L$ (or just $V \models CH + \diamondsuit(\omega_2)$), and $\aleph_3$ Cohen subsets of ω_1 are added. Then $\aleph_1$ is collapsed, and it is shown that, essentially, $\diamondsuit(\omega_2)^V$ can serve as a $\clubsuit$-sequence in the final model.

Subsequently J. Baumgartner in an unpublished note gave a different construction of a model of $\clubsuit + \neg CH$, in which $\aleph_1$ is not collapsed. P. Komjáth [Ko], continuing the proof in [Sh 98], proved it consistent to have MA for countable partial orderings $+\neg CH$, and $\clubsuit$. Then S. Fuchino, S. Shelah and L. Soukup [FShS 544] proved the same, without collapsing $\aleph_1$.

Having concluded that the principles, $\diamondsuit$ and $\clubsuit$, are different we still may ask to which extent the consequences of $\diamondsuit$ may be obtained from $\clubsuit$. So, I. Juhász asks in [Mi]: "Does $\clubsuit$ imply the existence of a Suslin tree?" This question is very natural, as $\diamondsuit$ was formulated by Jensen in [Je] in order to present his proof that there are Suslin trees in L. In addition, the existence of a Suslin tree is a long established test problem for various combinatorial principles to agree with.

Here we answer Juhász's question negatively.

The idea of the proof is to start with a model of $\diamondsuit + 2^{\aleph_1} = \aleph_2$, and iterate a forcing which specializes Suslin trees, in an iteration of length ω_2. Our plan is, similarly to [Sh 98], to witness $\clubsuit$ by using $\diamondsuit$ from the ground model in an essential way. Note that adding $\aleph_1$ Cohen reals destroys any club sequence from the ground model, which rules out finite support iterations.

Let χ be a large enough regular cardinal, and let $<^*_\chi$ be a fixed well order of $H(\chi)$. The formulation of $\diamondsuit$ that we use is that there is a sequence

$$\bar{N}^* = \langle \bar{N}^\delta = \langle N^\delta_i : i < \delta\rangle : \delta < \omega_1\rangle$$

where each $\bar{N}^\delta$ is a continuously increasing sequence of countable elementary submodels of $\mathfrak{A} \stackrel{\text{def}}{=} (H(\chi), \in, <^*_\chi)$ with $N^\delta_i \cap \omega_1 < \delta$ for $i < \delta$, and $\bar{N}^*$ is such that for every continuously increasing sequence $\bar{N} = \langle N_i \colon i < \omega_1\rangle$ of countable elementary submodels of $\mathfrak{A}$, there is a stationary set of $\delta < \omega_1$ such that the isomorphism type of $\bar{N}^\delta$ is the same as that of $\langle N_i : i < \delta \rangle$.

Let P denote our forcing order. To show that ♣ holds in V^P, we show that for every condition $p \in P$, name $\underset{\sim}{\tau}$ such that $p \Vdash$ "$\underset{\sim}{\tau} \in [\omega_1]^{\aleph_1}$", and a sequence $\bar{N}$ as above with $p, \underset{\sim}{\tau} \in N_0$, there is a club of $\delta < \omega_1$ for which there is an unbounded sequence $\bar{\beta}_{\bar{N} \restriction \delta} = \langle \beta_k : k < \omega \rangle \in V$ of ordinals below δ, and a condition $r^\oplus \geq p$ such that $r^\oplus \Vdash$ "$\{\beta_k : k < \omega\} \subseteq \underset{\sim}{\tau}$". Moreover, the choice of $\{\beta_k : k < \omega\}$ and the fact that $r^\oplus$ exists only depend on the isomorphism type of $\langle N_i : i < \delta \rangle$. Hence, if such a δ also has the property that the isomorphism type of $\langle N_i : i < \delta \rangle$ is the same as that of $\bar{N}^\delta$, then $\{\beta_k : k < \omega\}$ are definable from $\bar{N}^\delta$. So, the sequence $\langle A_\delta : \delta \text{ limit } < \omega_1 \rangle$ given by

$$A_\delta \stackrel{\text{def}}{=} \begin{cases} \mathrm{Rang}(\bar{\beta}_{\bar{N}^\delta}) & \text{if } \bar{\beta}_{\bar{N}^\delta} \text{ is defined} \\ \delta & \text{otherwise} \end{cases}$$

is a ♣-sequence in V^P. A typical consideration to make is the following. Suppose that p and $\underset{\sim}{\tau}$ are as above, while $\bar{N} = \langle N_n : n < \omega \rangle$ is an increasing sequence of countable $\prec \mathfrak{A}$, with $N_n \in N_{n+1}$ for $n < \omega$ and $P, p, \underset{\sim}{\tau} \in N_0$, and we wish to construct $\bar{\beta} \stackrel{\text{def}}{=} \bar{\beta}_{\bar{N}}$ and $r^\oplus$ as above. Let $N_\omega \stackrel{\text{def}}{=} \bigcup_{n<\omega} N_n$ and $\delta \stackrel{\text{def}}{=} N_\omega \cap \omega_1$. We can find $r^* \geq p$ and $\beta^* \in \underset{\sim}{\tau}$ such that $r^* \Vdash$ "$\beta^* \in \underset{\sim}{\tau}$". Now, we can reflect r^* and β^* along $\bar{N}$, and so obtain sequences $\langle r_n : n < \omega \rangle$ and $\langle \beta_n : n < \omega \rangle$ such that $r_n \Vdash$ "$\beta_n \in \underset{\sim}{\tau}$", while $\bigcup_{n<\omega} \beta_n = \delta$ and each $r_n \geq p$. If we can then find $r^\oplus$ as a common upper bound to $\{r_n : n < \omega\}$, we are done.

From what we said so far, our concerns are twofold: to have a forcing in which a certain amount of completeness is present, and, on the other hand, to have a control of the way the reals are added (of course, we need to add reals, as we need to violate $\diamondsuit$). In the direction of our second aim, we divide the iteration in $EVEN$ and ODD stages, and at the $EVEN$ stages we add a real which dominates all the reals in the previous model. In ODD stages we do a forcing $NNR(T)$ which specializes an Aronszajn tree T, doing so without adding reals. Our forcing at ODD stages is from S. Shelah's [Sh -f, V §6]. At $EVEN$ stages, we use the forcing UM for adding a universal meager set introduced by J. Truss in [Tr] (there it was called "amoeba forcing for category"), and used in S. Shelah's [Sh 176]. This forcing adds a dominating real. The forcing is *ccc* in a strong way, and, as shown by Shelah in [Sh 176], it has a strong "completeness" property, so-called sweetness, which guarantees that many ω-sequences of conditions have an upper bound. This in particular implies that there is a dense set $\mathcal{D}$ of conditions in UM on which there are equivalence relations $\langle E_n\colon n<\omega\rangle$, such that if a sequence $\bar{p}=\langle p^n\colon n\leq\omega\rangle$ from $\mathcal{D}$ has the property that $p^n\,E_n\,p^\omega$ for all n, then there is an upper bound to $\bar{p}$. A forcing notion with this property, expanded by such $\langle\langle\mathcal{D},E_n\colon n<\omega\rangle\rangle$, is called a sweetness model (see §2 for a better definition, and [Sh 176] for a real discussion). For a more recent application of these ideas, see [RoSh 672]. Our problem with completeness is then addressed by the way the iteration is done: we iterate with countable supports, but allow a condition p_1 to extend a condition p_0 only if the set of $EVEN$ coordinates in the $\mathrm{Dom}(p_0)$ on which p_1 differs from p_0 is finite (see [Sh -f, XIV] for a general treatment of such iterations and further references; an example of such an iteration used in connection with ♣ is in [DjSh574]). Basically, because at our $EVEN$ stages we are doing a *ccc* forcing, and adding a dominating real, we can afford to do such an iteration and still end up with a proper forcing.

Now consider again p and r^* from our above-described scenario. Before choosing r^*, we can construct increasing sequences $\bar{p}=\langle p_n\colon n<\omega\rangle$ and $\langle q_n\colon n<\omega\rangle$ which are sufficiently generic, in the following sense. We start with $p_0=p$, and choose p_n and q_n by induction on n. We shall have that p_{n+1} and p_n agree on $EVEN$ coordinates (we say $p_n\leq_{\mathrm{pr}}p_{n+1}$), while $q_n\geq p_n$ and they agree on the ODD coordinates, and $\mathrm{Dom}(p_n)=\mathrm{Dom}(q_n)$ (we say $p_n\leq_{\mathrm{apr}}q_n$). During the induction, we make sure that for every formula φ with parameters in N_ω, there are infinitely many n such that, given p_n, if we could have chosen p_{n+1} and q_{n+1} so that $\varphi(p_{n+1},q_{n+1})$ holds and the above description is not violated, then we have done so. We can also make sure that p_n's don't increase too much (for this we need to use dominating reals added by UM's along the way, and the way the iteration is defined), and thanks to a completeness-style property of $NNR(T)$-forcing this allows us, at the end of this induction, to define p_ω as the limit of all p_n. Now we can take $r^*\geq p_\omega$. We can find a condition p^* such that $p_\omega\leq_{\mathrm{pr}}p^*\leq_{\mathrm{apr}}r^*$, and we can arrange that the only odd coordinates on which p^* and r^* differ are those in $\mathrm{Dom}(p_\omega)$.

The set v_0 of $EVEN$ coordinates in the domain of p_ω where r^* and p_ω differ is finite, so is contained in N_{n_0} for some $n_0<\omega$. The idea now is that $\langle r_k\colon k<\omega\rangle$ will be a subsequence $\langle q_{n_k}\colon k<\omega\rangle$ of $\langle q_n\colon n<\omega\rangle$, constructed by induction on k. By exhibiting at every stage k a suitable formula φ_k with parameters in N_ω, such that $\varphi(x,y)$ densely holds for $x\geq_{\mathrm{pr}}p_{n_k}$ and $y\geq_{\mathrm{apr}}x$, we shall be able to control various properties of r_k's. For example, we'll be able to say that $q_{n_{k+1}}\Vdash$ "$\beta_{k+1}\in\underset{\sim}{\tau}$" for some $\beta_{k+1}\geq N_{n_k}\cap\omega_1$. Due to the nature of the $NNR(T)$ forcing and the preparations we made so far, we reduce the problem of

$\langle r_k\colon k < \omega\rangle$ having an upper bound, basically to the problem of the projections of q_k's onto v_0 having an upper bound. However, this is not exactly what happens, because these projections are not necessarily conditions in P.

Given any condition x in P, if we consider all the conditions y such that $y \geq_{\rm apr} x$, we obtain a sweetness model, R_x (we really use a variation called R_x^+). We shall aim at a condition $r' \in R_{p_\omega}$ such that $q_{n_k}\, E_k\, r'$, where E_k stands for the k-th equivalence relation in the sweetness model $R_{p\omega}$. Then we can use sweetness to assure that there is an upper bound as desired. What do we use as r'?

The condition we would really like to use as r' is $r^* \restriction \mathrm{Dom}(p_\omega)$. However, there are possibly coordinates of r^* which are less than $\sup(N_\omega \cap \omega_2)$ and not in N_ω, and names of $r^*(\gamma)$ for $\gamma \in \mathrm{Dom}(p_\omega)$ might depend on these "ghost" coordinates. So $r^* \restriction \mathrm{Dom}(p_\omega)$ might not be a condition after all. Hence we have a task of finding a $r' \in R_{p_\omega}$ which resembles r^* sufficiently, and let r_k's be more and more equivalent to this r'. However, we also have to be sure that our r_k's will be able to say something about β^*, to deliver the goods we implored them for.

We now place the entire situation in another countable elementary submodel of $\mathfrak{A}$, called M. We construct an increasing sequence $\bar{s} = \langle s_n\colon n < \omega\rangle$ sufficiently generic for M, starting with $s_0 = p^*$, and requiring s_n to only differ from p^* on the coordinates outside of N_ω. We let r' be whatever $\bar{s}$ forces r^* to be inside of the $\mathrm{Dom}(p_\omega)$, i.e. $r' = r^*/\bar{s}$ (see §8 for a more precise definition). As s_n's were chosen to be sufficiently generic, we'll have that the naturally defined join of s_n and r' will have the same n-th equivalence class in R_{p^*} as r' does in R_{p_ω}, for all large enough n. In §6 we develop a method of saying this through a first order formula. Note that this join still contains the relevant information about β^*. So, using again the genericity of $\langle p_n\colon n < \omega\rangle$ we are done.

Swept under the rug in the above discussion is the fact that all the choices that we make have to be made depending only on the isomorphism type on $\bar{N}$, but this is easily arranged thanks to the well ordering of $H(\chi)$.

Taken with a grain of salt, as no proofs were given of our claims so far, and as introductions are usually easier to understand once whatever they are supposed to introduce is already understood, the above explanation might have convinced the reader that what we do is sufficient to prove the desired theorem. But is all this machinery really necessary? We can only say that we tried several other approaches, and the difficulty that we would face in general is that some amount of completeness was missing. Such completeness in the present proof is achieved through the use of sweetness. One could presumably obtain a simpler proof that *some* different version of ♣ does not imply the existence of a Suslin tree.

Saharon Shelah has notes in which a version of the order from [BMR] is iterated with supports similar to the ones we are using, and the iteration shows that a weak version of $\clubsuit$ does not imply the existence of a Suslin tree. However, by the results in [DjSh574], this version of $\clubsuit$ is strictly weaker than $\clubsuit$.

The paper is organized as follows. In §2 we give some background to $NNR(T)$ and UM. In §3 we state the Theorem. In §4 we describe the iteration and prove various facts about it. The proof of the properness of the forcing used is contained in §5. In §6 we give some definitions which are used to adapt the notion of sweetness to our situation. However, the notions from [Sh 176] have to be reformulated to fit our needs, hence in particular our discussion is completely self-contained. In §7 we introduce some auxiliary partial orders, and set the ground for the proof in §8. The main point of the proof is to obtain $\clubsuit$ in V^P. This argument is presented in §8.

2. Background

The forcing needed in §3 will be an iteration of two kinds of individual forcings. The first is the forcing from [Sh -f, V§6], which specializes an Aronszajn tree T without adding reals. We shall refer to this forcing as $NNR(T)$. The other individual forcing is UM ("universal meager"), the forcing introduced in [Tr] and used in [Sh 176, §7]. In this section we review some properties of these forcings that will be needed for the proof in §3–§8.

Notation 2.1: (1) For two sequences $\bar{s}$ and $\bar{t}$, we say that $\bar{s}\cap\bar{t}=\emptyset$ whenever the ranges of $\bar{s}$ and $\bar{t}$ are disjoint.
(2) $\mathbb{Q}$ stands for the rational numbers with their usual ordering.
(3) If T is a tree, then $<_T$ denotes the tree order of T. For $x \in T$, we let $\mathrm{ht}_T(x) \stackrel{\mathrm{def}}{=} \mathrm{otp}(\{y\colon y <_T x\})$. We may omit T in this notation, if the T we mean is clear from the context.

If T is an ω_1-tree and $i < \omega_1$, then T_i denotes the i-level of T, i.e. $\{x \in T\colon \mathrm{ht}(x) = i\}$.

If $\bar{x}$ and $\bar{y}$ are two sequences of elements of T, then $\bar{x} <_T \bar{y}$ means that $\bar{x}$ and $\bar{y}$ have the same length and, for every $l \in \mathrm{Dom}(x)$, we have $x_l <_T y_l$.

If $\mathrm{Rang}(\bar{x}) \cap \mathrm{Rang}(\bar{y}) = \emptyset$, we say that $\bar{x}$ and $\bar{y}$ are **disjoint**.
(4) If η and ρ are sequences, then $\eta \lhd \rho$ means that η is an initial segment of ρ.
(5) Without loss of generality, all Aronszajn trees T that we mention will be assumed to have the property that $T_\alpha \subseteq [\omega\alpha, \omega(\alpha+1))$, for all $\alpha < \omega_1$. In addition, we'll assume $|T_\alpha| \leq \aleph_0$ for all $\alpha < \omega_1$. As we might want to consider subtrees of T, we do not assume necessarily that $T_\alpha = [\omega\alpha, \omega(\alpha+1))$ for all α.

(6) If T is an Aronszajn tree and $\alpha < \omega_1$, then $\{x_l^{T_\alpha}: l < l^*(\alpha) \leq \omega\}$ is the increasing enumeration of T_α.
(7) Suppose that T is an Aronszajn tree and $m < \omega$, while $\alpha < \omega_1$. We define

$$w_m^{T_\alpha} \stackrel{\text{def}}{=} \{x_l^{T_\alpha}: l < m\}.$$

(8) We often identify a node $x \in T_\delta$ for limit δ with the branch $\{y: y <_T x\}$. Also, if $\alpha < \beta$ and $x \in T_\beta$, then $x \restriction (\alpha+1)$ denotes the unique $y \in T_\alpha$ with $y <_T x$.

Definition 2.2: Given an Aronszajn tree T, we define

$$\begin{aligned}(1)\quad NNR_1(T) \stackrel{\text{def}}{=} \{(f,C)\colon\ & C \text{ is a closed subset of some } \alpha+1<\omega_1\\ & \text{with the last element } \alpha \stackrel{\text{def}}{=} \mathrm{lt}(C), \text{ and}\\ & f\colon \textstyle\bigcup_{i\in C} T_i \to \mathbb{Q} \text{ is monotonically increasing}\}.\end{aligned}$$

For (f_1, C_1) and (f_2, C_2) in $NNR_1(T)$, we say $(f_1,C_1) \leq_{NNR_1(T)} (f_2,C_2)$ iff $C_1 = C_2 \restriction (\mathrm{lt}(C_1)+1)$ and $f_1 \subseteq f_2$.
(2) Γ is a T-**promise** iff there is a club $C(\Gamma)$ of ω_1 and $n = n(\Gamma) < \omega$ such that:
(a) All elements of Γ have form $\langle x_0, \ldots, x_{n-1}\rangle$ where $\langle x_0, \ldots, x_{n-1}\rangle$ are such that

$$(\exists \alpha \in C(\Gamma))[(\forall i \neq j < n)\,(x_i \neq x_j)\ \&\ (\forall i < n)\,(x_i \in T_\alpha)].$$

(b) If $\alpha < \beta \in C(\Gamma)\ \&\ \bar{x} \in \Gamma \bigcap {}^n T_\alpha$, then there are infinitely many pairwise disjoint $\bar{y} \in {}^n T_\beta$ such that $\bar{x} <_T \bar{y}$.
(c) $\Gamma \bigcap {}^n(T_{\min(C(\Gamma))}) \neq \emptyset$.
(3) $(f,C) \in NNR_1(T)$ **fulfills a promise** Γ iff
(α) $\mathrm{lt}(C) \in C(\Gamma)$ and $C(\Gamma) \supseteq C \setminus \min(C(\Gamma))$.
(β) For all $\alpha < \beta \in C(\Gamma) \bigcap C$, and for all $\bar{x} \in \Gamma \bigcap {}^{n(\Gamma)}(T_\alpha)$ the following holds:
($\oplus$) For all $\epsilon > 0$, there are infinitely many pairwise disjoint $\bar{y} \in {}^{n(\Gamma)}T_\beta$ with $\bar{x} <_T \bar{y}$ and such that for all $l < n(\Gamma)$ we have

$$f(x_l) < f(y_l) < f(x_l) + \epsilon.$$

The intention of fulfilling a promise is that f is guaranteed not to grow too much along the relevant branches.

$$\begin{aligned}(4)\quad NNR(T) \stackrel{\text{def}}{=} \{(f,C,\Psi) : (f,C) \in NNR_1(T) & \text{ and } \Psi \text{ is a countable set}\\ & \text{of promises which } (f,C) \text{ fulfills}\}.\end{aligned}$$

We let $(f_1, C_1, \Psi_1) \leq (f_2, C_2, \Psi_2)$ iff $(f_1,C_1) \leq_{NNR_1(T)} (f_2,C_2)$ and Ψ_1 is a subset of Ψ_2, while $C_2 \setminus C_1 \subseteq \bigcap_{\Gamma\in\Psi_1} C(\Gamma)$.

Notation 2.3: For $p = (f, C, \Psi) \in NNR(T)$, we write $f^p \stackrel{\text{def}}{=} f$, $C^p \stackrel{\text{def}}{=} C$, $\Psi^p = \Psi$ and $\mathrm{lt}(p) \stackrel{\text{def}}{=} \mathrm{lt}(C_p)$.

Definition 2.4 [Sh -f, VIII §2]: Given κ an infinite cardinal, a forcing notion P is said to satisfy **κ-pic*** iff for all large enough χ and well orders $<^*_\chi$ of $H(\chi)$, we have:

Suppose that $i < j < \kappa$, and $N_i, N_j \prec \mathfrak{A} = (H(\chi), \in, <^*_\chi)$ are countable with $\kappa, P \in N_i \cap N_j$, while $N_i \cap \kappa \subseteq j$ and $N_i \cap i = N_j \cap j$, and N_l is the Skolem hull in $\mathfrak{A}$ of $(N_i \bigcap N_j) \cup \{l\}$ for $l \in \{i, j\}$. Further suppose that $p \in P \cap N_i$, while $h\colon N_i \to N_j$ is an isomorphism with $h \restriction (N_i \cap N_j)$ being the identity, and $h(i) = j$.

Then there is $q \in P$ such that

(a) $p, h(p) \leq q$, and for every maximal antichain $I \subseteq P$ with $I \in N_i$, we have that $I \cap N_i$ is predense above q.

(b) For every $r \in N_i \cap P$ and q' such that $q \leq q' \in P$, there is $q'' \in P$ such that
$$r \leq q' \quad \text{iff } h(r) \leq q''.$$

FACT 2.5 [Sh -f, VIII, 2.5* and 2.9*]: Suppose that $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha \colon \alpha < \alpha^* \rangle$ is a countable support iteration and κ is regular. Further suppose that for each $\alpha < \alpha^*$ we have $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ has κ-pic*." Then :

(1) If $\alpha^* < \kappa$, then P_{α^*} satisfies κ-pic*.

(2) If $\alpha^* \leq \kappa$ and $(\forall \mu < \kappa)\, (\mu^{\aleph_0} < \kappa)$, then P_κ satisfies $\kappa - cc$.

(3) If $\alpha^* < \kappa$ and $(\forall \mu < \kappa)\, (\mu^{\aleph_0} < \kappa)$, then
$$\Vdash_{P_{\alpha^*}} \text{“}(\forall \mu < \kappa)\, (\mu^{\aleph_0} < \kappa)\text{”}.$$

FACT 2.6 [Sh -f, V§6]: Suppose $V \models CH$. Then $NNR(T)$ is a proper $\aleph_2$-*cc*, moreover $\aleph_2$-pic*, forcing which specializes T without adding reals.

Note that $|NNR(T)| \leq 2^{\aleph_1}$.

FACT 2.7 [Sh -f V, 6.7]: Suppose that χ is large enough and $N \prec (H(\chi), \in)$ is countable such that $T \in N$. Let $\delta \stackrel{\text{def}}{=} N \cap \omega_1$ and $\epsilon > 0$. Further suppose that $p \in NNR(T) \cap N$ and for some $n < \omega$ we have $b_0, \ldots, b_{n-1}$ are distinct branches of T_δ, while $I \in N$ is an open dense subset of P.

Then there is $q \geq p$ with $q \in I \cap N$, and such that for all $i < n$ we have
$$f^q(b_i(\mathrm{lt}(q))) < f^p(b_i(\mathrm{lt}(p))) + \epsilon.$$

The following is well known and follows from the above Fact 2.7:

CLAIM 2.8: *Suppose that χ is large enough and $N \prec (H(\chi), \in)$ is countable such that $T \in N$, while $p \in NNR(T) \cap N$. Then there is $q \geq p$ which is $(N, NNR(T))$-generic and* $\mathrm{lt}(q) = N \cap \omega_1$.

Proof of the Claim: Let $\{I_n \colon n < \omega\}$ enumerate all open dense subsets of $NNR(T)$ which are elements of N. Using Fact 2.7, we can build an increasing sequence $\langle p_n \colon n < \omega \rangle$ of conditions in $NNR(T)$ such that:

(a) $p_0 = p$.

(b) $p_n \in N$.

(c) For every $n < \omega$, for every $x \in w_{n+1}^{T_{\mathrm{lt}(p_{n+1})}}$ we have

$$f^{p_{n+1}}(x) < f^{p_n}(x \restriction [\mathrm{lt}(p_n) + 1]) + 1/2^n.$$

(d) $p_{n+1} \in I_n$.

Now we can define q by letting $\mathrm{lt}(q) \stackrel{\mathrm{def}}{=} \delta$, $C^q \stackrel{\mathrm{def}}{=} \bigcup_{n<\omega} C^{p_n} \cup \{\delta\}$, while

$$f^q \stackrel{\mathrm{def}}{=} \bigcup_{n<\omega} f^{p_n} \cup \{(x, \sup_{n<\omega}(f^{p_n}(x \restriction [\mathrm{lt}(p_n)+1])) \colon \\ x \in T_\delta \& x \restriction [\mathrm{lt}(p_n)+1] \in \mathrm{Dom}(f^{p_n})\},$$

and $\Psi^q \stackrel{\mathrm{def}}{=} \bigcup_{n<\omega} \Psi^{p_n}$. It is easily seen that q is as required. $\blacksquare_{2.8}$

Definition 2.9: (1) $\mathcal{T} \subseteq {}^{<\omega}2$ is a **nowhere dense tree** iff , for all $\eta \in \mathcal{T}$, there is $\rho \in {}^{<\omega}2 \setminus \mathcal{T}$ with $\eta \lhd \rho$.

(2) $\mathcal{T} \subseteq {}^{<\omega}2$ is **perfect** iff, for all $\eta \in \mathcal{T}$, there are $\rho_1 \neq \rho_2$ both in $\mathcal{T}$ and both extending η.

(3) $$UM \stackrel{\mathrm{def}}{=} \{(t, \mathcal{T}) \colon \mathcal{T} \subseteq {}^{<\omega}2 \text{ is a perfect nowhere dense tree} \\ \text{and for some } n \text{ we have } t = \mathcal{T} \cap {}^n2\}.$$

For $(t_1, \mathcal{T}_1), (t_2, \mathcal{T}_2) \in UM$, we say $(t_1, \mathcal{T}_1) \leq (t_2, \mathcal{T}_2)$ iff for some n we have $t_1 = t_2 \cap {}^n2$, and $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

FACT 2.10 [Tr]: Suppose that G is UM-generic.
Then $S \stackrel{\mathrm{def}}{=} \bigcup\{\mathcal{T} \colon (\exists t)((t, \mathcal{T}) \in G)\}$ is a nowhere dense subtree of ${}^{<\omega}2$.

The following consequence of Fact 2.10 is also well known:

CLAIM 2.11: *UM adds a real which dominates all the reals from the ground model.*

Proof of the Claim: Let S be the nowhere dense tree added by UM. We define $g_S \in {}^\omega\omega$ by letting

$$g_S(n) \stackrel{\mathrm{def}}{=} \min\{m \colon (\forall \eta \in S \cap {}^n2)(\exists \rho \in {}^m2 \setminus S)(\eta \lhd \rho)\}.$$

Hence g_S is well defined, and we shall now see that it dominates all $f \in {}^\omega\omega$ of the ground model. Fix such an f, and note that the set of all conditions $(t, \mathcal{T})$ which satisfy

$$(\exists n_0)(\forall n \geq n_0)\,[\min\{m\colon (\forall \eta \in {}^n2)(\exists \rho \in {}^m2 \setminus \mathcal{T})\,(\eta \lhd \rho)\} > f(n)]$$

is dense in UM. $\blacksquare_{2.11}$

Notation 2.12: For $p = (t, \mathcal{T}) \in UM$, we let $t^p \stackrel{\text{def}}{=} t$ and $\mathcal{T}^p \stackrel{\text{def}}{=} \mathcal{T}$.

Definition 2.13 [Sh 176, §7]*:* (1) A forcing notion P is **sweet** if there is a subset of $\mathcal{D}$ of P and equivalence relations E_n on $\mathcal{D}$ for $n < \omega$, such that:

(a) $\mathcal{D} \subseteq P$ is dense, E_{n+1} refines E_n and E_n has countably many equivalence classes.

(b) For every $n < \omega$ and $p \in \mathcal{D}$, the equivalence class p/E_n is directed.

(c) If $p^i \in \mathcal{D}$ for $i \leq \omega$, and $p^i\, E_i\, p^\omega$, then $\{p^i\colon i \leq \omega\}$ has an upper bound; moreover, for each $n < \omega$ the set $\{p^i\colon n \leq i \leq \omega\}$ has an upper bound in p^ω/E_n.

(d) For every p, q in $\mathcal{D}$ and $n < \omega$, there is $k < \omega$ such that for every $p' \in p/E_k$,

$$(\exists r \in q/E_n)\,(r \geq p) \implies (\exists r \in q/E_n)\,(r \geq p').$$

(2) If (1) above holds, we say that $(P, \mathcal{D}, E_n)_{n<\omega}$ is a **sweetness model**.

Definition 2.14 [Sh 176, §7]*:* Suppose that $\mathfrak{B} = (P, \mathcal{D}^0, E^0_n)_{n<\omega}$ is a sweetness model and $\bar{A} = \langle A_e\colon e < \omega\rangle$ enumerates $\{p/E^0_n\colon n < \omega \,\&\, p \in \mathcal{D}^0\}$.

(1) For $q \in \mathcal{D}^0$ we define $k_m(q)$ as the minimal $k < \omega$ such that for every $q' \in q/E^0_k$ we have that

$$(\exists r \in A_m)\,(r \geq q) \quad \text{iff } (\exists r' \in A_m)\,(r' \geq q').$$

(2) We define

$$\mathcal{D} \stackrel{\text{def}}{=} \{(p, (t, \underset{\sim}{\mathcal{T}}))\colon p \in \mathcal{D}^0 \;\&\; \Vdash_P \text{“}(t, \underset{\sim}{\mathcal{T}})\text{”} \in \underset{\sim}{UM}\text{”}\}.$$

For $n < \omega$ and $(p_l, (t_l, \underset{\sim}{\mathcal{T}}_l)) \in P * \underset{\sim}{UM}$ $(l = 1, 2)$, we say that

$$(p_1, (t_1, \underset{\sim}{\mathcal{T}}_1))\, E_n\, (p_2, (t_2, \underset{\sim}{\mathcal{T}}_2))$$

iff the following conditions hold:

(α) $p_1\, E^0_n p_2$,

(β) $t_1 = t_2$,

(γ) for every $m < n$, there is $p \in A_m$ with $p \geq p_1$ iff there is $p \in A_m$ with $p \geq p_2$,

(δ) suppose that $m < n$ and there is $p \in A_m$ such that $p \geq p_1$, and let $\eta \in {}^{<n}2$; then there is $p \in A_m$ such that $p \Vdash$ "$\eta \notin \underset{\sim}{\mathcal{T}}_1$" iff there is $p \in A_m$ such that $p \Vdash$ "$\eta \notin \underset{\sim}{\mathcal{T}}_2$",

(ε) for all $m < n$ we have $k_m(p_1) = k_m(p_2)$ and for all $m < n$ we have $p_1 \, E^0_{k_m(p_1)} \, p_2$.

Definition 2.15 [Sh 176, §7]: Sweetness models $\mathfrak{B}_1 \stackrel{\text{def}}{=} (P^1, \mathcal{D}^1, E^1_n)_{n<\omega}$ and $\mathfrak{B}_2 \stackrel{\text{def}}{=} (P^2, \mathcal{D}^2, E^2_n)_{n<\omega}$ are said to satisfy $\mathfrak{B}_1 < \mathfrak{B}_2$ iff:

(a) P^1 is a complete suborder of P^2, while $\mathcal{D}^1 \subseteq \mathcal{D}^2$ and for each n we have that E^1_n is E^2_n restricted to $\mathcal{D}^1$.

(b) For all $p \in \mathcal{D}^1$ and $n < \omega$ we have $p/E^2_n \subseteq P^1$.

(c) If $p \leq q$ and $q \in \mathcal{D}^1$, while $p \in \mathcal{D}^2$, then $p \in \mathcal{D}^1$.

Notation 2.16: Suppose that $\mathfrak{B}$, $\bar{A}$ and P are as in the assumptions of Definition 2.14 and $\mathcal{D}$ and E_n ($n < \omega$) are as defined in Definition 2.14. We say that

$$\mathfrak{B}_{\bar{A}} * \underset{\sim}{UM} \stackrel{\text{def}}{=} (P * \underset{\sim}{UM}, \mathcal{D}, E_n)_{n<\omega}$$

is **the canonical sweetness model** on $P * \underset{\sim}{UM}$ **with respect to $\mathfrak{B}$ and $\bar{A}$.**

FACT 2.17 [The Composition Claim, Sh 176, §7]: If $\mathfrak{B}$ is a sweetness model and $\bar{A}$ is an enumeration of the equivalence classes of $\mathfrak{B}$, then $\mathfrak{B}_{\bar{A}} * \underset{\sim}{UM}$ is a sweetness model and $\mathfrak{B} < \mathfrak{B}_{\bar{A}} * \underset{\sim}{UM}$.

FACT 2.18 [Sh 176, §7]: Suppose that for $k < n$ we have that $(P^k, \mathcal{D}^k, E^k_n)_{n<\omega}$ is a sweetness model and

$$(P^k, \mathcal{D}^k, E^k_n)_{n<\omega} < (P^{k+1}, \mathcal{D}^{k+1}, E^{k+1}_n)_{n<\omega}.$$

Then $(\bigcup_{k<\omega} P^k, \bigcup_{k<\omega} \mathcal{D}^k, \bigcup_{k<\omega} E^k_n)_{n<\omega}$ is a sweetness model with the property that for all $k < \omega$ we have

$$(P^k, \mathcal{D}^k, E^k_n)_{n<\omega} < (\bigcup_{k<\omega} P^k, \bigcup_{k<\omega} \mathcal{D}^k, \bigcup_{k<\omega} E^k_n)_{n<\omega}.$$

NOTE 2.19: Any sweet forcing P is *ccc*, even is σ-centered.

[Why? Let $\{A_m \colon m < \omega\}$ enumerate all q/E_n for $q \in \mathcal{D}$ and $n < \omega$. For $m < \omega$, let $B_m \stackrel{\text{def}}{=} \{q : (\exists p \in A_m)\,(p \geq q)\}$, hence each B_m is directed and $P = \bigcup_{m<\omega} B_m$.]

3. ♣ does not imply the existence of a Suslin tree

THEOREM 3.1: *Assume that* $V \models$ "$\diamondsuit(\omega_1) + 2^{\aleph_1} = \aleph_2$".

Then *there is a proper* $\aleph_2 - cc$ *forcing notion* P *such that* $\Vdash_P$ "$\clubsuit+$ *there are no Suslin trees*".

The proof of this Theorem is presented in §4–§8.

4. Forcing and iteration

NOTATION 4.1: "$\mathcal{T}$ is NWD" means that $\mathcal{T}$ is a perfect nowhere dense subtree of ${}^{<\omega}2$.

Definition 4.2: By simultaneous induction on $\alpha \leq \omega_2$, we define items (1)–(3) and prove Claim 4.3 below.

(1)

$$P_\alpha \stackrel{\text{def}}{=} \left\{ p : \begin{array}{l} (I)\ \mathrm{Dom}(p) \text{ is a countable } \subseteq \alpha \\ (II)\ \text{For all } i \in \mathrm{Dom}(p) \text{ we have} \\ \qquad \Vdash_{P_i} \text{“}p(i) \in \underset{\sim}{Q}_i\text{”} \end{array} \right\}.$$

(2) If $\alpha = 2i$ for some i, then $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha = \underset{\sim}{UM}$".

(3) If $\alpha = 2i+1$ for some i, then $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha = NNR(\underset{\sim}{\mathcal{T}}^\alpha)$", where $\underset{\sim}{\mathcal{T}}^\alpha$ is a P_{2i}-name of an Aronszajn tree, handed to us by the bookkeeping (see Claim 4.3 below). (We **emphasize** that $\underset{\sim}{\mathcal{T}}^\alpha$ is a P_{2i}-name, not a P_α-name.)

(4) We say that $p \leq q$ for $p, q \in P_\alpha$ iff for all $j \in \mathrm{Dom}(p)$ we have:
   (a) j even $\Longrightarrow q \restriction j \Vdash_{P_j}$ "$p(j) \leq q(j)$" and
   (b) $\{2i \in \mathrm{Dom}(p) : \neg(q \restriction (2i) \Vdash \text{“}p(2i) = q(2i)\text{”})\}$ is finite.
   (c) j odd $\Longrightarrow \Vdash_{P_j}$ "$p(j) \leq q(j)$". ∎

(Note that $(P_\alpha, \leq)$ is a forcing notion.)

CLAIM 4.3: *If* $\alpha = 2i+1$ *for some* i, then

$$\Vdash_{P_\alpha} \text{“}\underset{\sim}{\mathcal{T}}^\alpha \textit{ is an Aronszajn tree”}.$$

Proof of the Claim: This easily follows from the fact that UM is σ-centered (Fact 2.17 and Note 2.19, and even the property of Knaster suffices). Namely, suppose that $\alpha = 2i+1$. We know that

$$\Vdash_{P_{2i}} \text{“}\underset{\sim}{\mathcal{T}}^\alpha \text{ is an Aronszajn tree”},$$

as $\underset{\sim}{T}^\alpha$ is a P_{2i}-name of an Aronszajn tree. We only have to check that $\underset{\sim}{Q}_{2i}$ does not add any uncountable branches to $\underset{\sim}{T}^\alpha$. We work in $V^{P_{2i}}$. Suppose $p \in UM$ and

$$p \Vdash \text{“} \underset{\sim}{\tau} : \omega_1 \to T^\alpha \text{ is increasing } \& \ (\forall \gamma < \omega_1)(\underset{\sim}{\tau}(\gamma) \in T^\alpha_\gamma)\text{”}.$$

For $\gamma < \omega_1$, let $\mathcal{D}_\gamma$ be the set of conditions of UM which are above p and decide the value of $\underset{\sim}{\tau}(\gamma)$. Then there is a directed subset A of UM and an uncountable $B \subseteq \omega_1$ such that

$$\gamma \in B \Longrightarrow A \cap \mathcal{D}_\gamma \neq \emptyset.$$

It follows from the directedness of A that, for all $\gamma \in B$, there is a unique l_γ such that for all $q \in \mathcal{D}_\gamma \cap A$ we have $q \Vdash$ “$\underset{\sim}{\tau}(\gamma) = x^{T^\alpha_\gamma}_{l_\gamma}$”. Again by the directedness of A, if $\gamma_1 < \gamma_2 \in B$ we must have $x^{T^\alpha_{\gamma_1}}_{l_{\gamma_1}} <_{T^\alpha} x^{T^\alpha_{\gamma_2}}_{l_{\gamma_2}}$, a contradiction. ■4.3

NOTATION 4.4: (1) For $j < \alpha$ and $p, r \in P_j$ we say $p \leq_{\text{apr}} r$ iff
 (i) $\mathrm{Dom}(p) = \mathrm{Dom}(r)$ and
 (ii) $p \leq r$ and
 (iii) $(\forall 2i+1 \in \mathrm{Dom}(p))\, (r \restriction (2i+1) \Vdash_{P_{2i+1}}$ “$r(2i+1) = p(2i+1)$”$)$.
(2) For $j < \alpha$ and $p, r \in P_j$ we say $p \leq_{\text{pr}} r$ iff
 (a) $p \leq r$ and
 (b) $(\forall 2i \in \mathrm{Dom}(p))\, (r \restriction (2i) \Vdash_{P_{2i}}$ “$r(2i) = p(2i)$”$)$.

Observation 4.5: $\leq_{\text{pr}}$ and $\leq_{\text{apr}}$ are partial orders.

Definition 4.6: By simultaneous induction on $\alpha \leq \omega_2$, we define items (1)–(4) below and prove Claim 4.7 below.*

$$(1) \qquad P'_\alpha \overset{\text{def}}{=} \left\{ p \in P'_\alpha : \begin{array}{l} (A)\ \text{If } 2i \in \mathrm{Dom}(p), \text{then} \\ \quad p(2i) \text{ is simple above } p \restriction 2i \\ (B)\ \text{There is } \delta^*(p) \text{ limit } < \omega_1 \text{ such that} \\ \quad 2i+1 \in \mathrm{Dom}(p) \Longrightarrow \\ \quad \Vdash_{P_{2i+1}} \text{“}\mathrm{lt}(p(2i+1)) = \delta^*(p)\text{”} \end{array} \right\},$$

with the order inherited from P_α.

(2) If $2i < \alpha$ and $\underset{\sim}{\tau} = (\underset{\sim}{t}^{\tau}, \underset{\sim}{T}^{\tau})$ is a P_{2i}-name for a condition in $\underset{\sim}{UM}$, while $q \in P_{2i}$, we say that q **determines** $\underset{\sim}{\tau}$ **to degree** n iff
 (i) q forces in P_{2i} a value to $\underset{\sim}{T}^{\tau} \cap {}^{\leq n}2$,

* Later we shall prove that P'_α is a dense subset of P_α, for all $\alpha \leq \omega_2$.

(ii) q forces in P_{2i} a value to $\underset{\sim}{t}^{\underset{\sim}{\mathcal{T}}}$,

(iii) for all $\eta \in {}^{\leq n}2$, there is $\nu \rhd \eta$ such that

$$q \Vdash_{P_{2i}} \text{“}\nu \notin \underset{\sim}{\mathcal{T}}^{\underset{\sim}{\mathcal{T}}}\text{”},$$

(iv) for all $\eta \in {}^{\leq n}2$, there are $\eta_1 \neq \eta_2 \rhd \eta$ such that

$$q \Vdash_{P_{2i}} \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{\underset{\sim}{\mathcal{T}}} \Longrightarrow \eta_1, \eta_2 \in \underset{\sim}{\mathcal{T}}^{\underset{\sim}{\mathcal{T}}}\text{”}.$$

(3) For $p \in P_\alpha$ we define

$$R_p \stackrel{\text{def}}{=} \{q \in P'_\alpha : q \geq_{\text{apr}} p\},$$

with the order inherited from P'_α.

(4) For $2i < \alpha$ and $p \in P_{2i}$, we say that $p(2i) = (\underset{\sim}{t}^{2i}, \underset{\sim}{\mathcal{T}}^{2i})$ is **simple above** $p \restriction 2i$ iff there are I_n $(n < \omega)$ such that

$$p \restriction 2i \Vdash_{P_{2i}} \text{“}(\forall n < \omega)[I_n \subseteq R_{(p\restriction 2i)} \text{ countable predense in } R_{p\restriction 2i} \;\&$$
$$(r \in I_n \Longrightarrow r \text{ determines } p(2i) \text{ to degree } n)]\text{”}.$$

Claim 4.7: *If $p \in P'_\alpha$ and $\beta < \alpha$, then $p \restriction \beta \in P'_\beta$.*

Proof of the Claim: *This is easily checked, noting that the definition of $p(2i)$ being simple above $p \restriction 2i$ only depends on $p \restriction 2i$, for $2i < \alpha$.* $\blacksquare_{4.7}$

Notation 4.8: (1) $p \geq_{\text{apr}} (\geq_{\text{pr}}, \geq)\, q$ iff $q \leq_{\text{apr}} (\leq_{\text{pr}}, \leq)\, p$.
(2) Let $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \omega_2, \beta < \omega_2\rangle$ and $\bar{Q}' = \langle P'_\alpha \colon \alpha \leq \omega_2\rangle$.
(3) $P \stackrel{\text{def}}{=} P'_{\omega_2}$.
(4) χ is a fixed large enough regular cardinal, and $<^*_\chi$ is a fixed well-ordering of $H(\chi)$.
(5) *EVEN* stands for the set of even ordinals, and *ODD* for the set of odd ones.
(6) Quantifier $\forall^*$ means “for all but finitely many”.
(7) $p \leq q \in P'_\alpha$ means that $p, q \in P'_\alpha$ and in P_α we have $p \leq q$.

Definition 4.9: Suppose that $2i < \omega_2$ and $p \in P_{2i}$, and $p(2i)$ is simple above $p \restriction 2i$, while $\bar{I} = \langle I_n \colon n < \omega\rangle$ are as in Definition 4.6(4). We say that $\bar{I}$ **exemplifies the simplicity of** $p(2i)$ **above** $p \restriction 2i$.

NOTE 4.10: (1) Suppose that $\alpha \leq \omega_2$ and $\langle \alpha_n : n < \omega \rangle$ is an increasing sequence of ordinals with $\sup_{n<\omega} \alpha_n = \alpha$. Further suppose that $\langle q_n : n < \omega \rangle$ is a sequence such that

(i) $q_n \in P_{\alpha_n}$ $[P'_{\alpha_n}]$ for all n,

(ii) $q_{n+1} \restriction \alpha_n = q_n$.

Then $q \stackrel{\text{def}}{=} \bigcup_{n<\omega} q_n$ is a condition in P_α $[P'_\alpha]$.

(2) For every $\alpha < \omega_2$ we have $\Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ is proper".

(3) If $\alpha \leq \omega_2$ and p, p' are such that $p \leq_{\text{apr}} p' \in P'_\alpha$, then

$$R_{p'} = \{q \in R_p : q \geq p'\}.$$

(4) If $p \leq_{\text{apr}} p' \in P'$, then $\delta^*(p) = \delta^*(p')$.

NOTATION 4.11: (1) Given $\gamma < \omega_2$ even, we let $\underset{\sim}{g}_\gamma$ be a P_γ-name for the dominating real added by $\underset{\sim}{Q}_\gamma$.

(2) Suppose that $\beta < \alpha \leq \omega_2$ and $A \subseteq P'_\alpha$. We define

$$A \restriction \beta \stackrel{\text{def}}{=} \{s \restriction \beta : s \in A\}.$$

(3) For $\alpha \leq \omega_2$ and $J \subseteq P'_\alpha$, we say that J is $\leq_{\text{pr}}$-**open** iff

$$(\forall p \in P'_\alpha)(\forall q \in J)(p \geq_{\text{pr}} q \Longrightarrow p \in J).$$

We say that J is $\leq_{\text{pr}}$-**dense above** $p \in P'_\alpha$ iff

$$(\forall q \in P'_\alpha)\,[q \geq_{\text{pr}} p \Longrightarrow (\exists r \in J)\,(r \geq_{\text{pr}} q)].$$

Observation 4.12: Suppose that $\beta < \alpha \leq \omega_2$ and $p \in P'_\alpha$. Further suppose that $J \subseteq P'_\alpha$ is $\leq_{\text{pr}}$-open and $\leq_{\text{pr}}$-dense above p.

Then $J \restriction \beta$ is $\leq_{\text{pr}}$-open $\leq_{\text{pr}}$-dense above $p \restriction \beta$.

Observation 4.13: Suppose that $\alpha \leq \omega_2$ and $p \leq q \in P_\alpha$, and let us define r as follows:

$$r(\beta) \stackrel{\text{def}}{=} \begin{cases} p(\beta) & \text{if } \beta \in (EVEN \cap \text{Dom}(p)), \\ q(\beta) & \text{otherwise,} \end{cases}$$

letting $\text{Dom}(r) = \text{Dom}(q)$. Then $r \in P_\alpha$ and r has the following properties:

(i) $p \leq_{\text{pr}} r \leq_{\text{apr}} q$.

(ii) $\neg(q \restriction \beta \Vdash$ "$r(\beta) = q(\beta)$"$) \Longrightarrow \beta \in \text{Dom}(p)$.

(iii) If there is δ^* such that for all odd $\beta \in \text{Dom}(q)$ we have $\Vdash_{P_\beta}$ "$\text{lt}(q) = \delta^*$, then for all such β we have $\Vdash_{P_\beta}$ "$\text{lt}(r(\beta)) = \delta^*$.

NOTATION 4.14: Suppose that $\alpha \leq \omega_2$ and $p \leq q \in P_\alpha$. Then r defined as in Observation 4.13 is denoted by $\mathrm{intr}(p,q)$.

CLAIM: *Given $\alpha \leq \omega_2$ and $p \in P'_\alpha$.* Then

$$p \Vdash_{P_\alpha} \text{“}R_p \textit{ is a ccc partial order”}.$$

(More is true; see Lemma 6.6.)

Proof of the Claim: By induction on α, for all $p \in P'_\alpha$ simultaneously. There are two eventful cases of the induction.

“$\alpha = \beta + 1, \beta$ even.” Note that $R_p \subseteq R_{p\restriction\beta} * \underset{\sim}{UM}$ is a suborder which is dense above p. (Or see the proof of Claim 5.1 $(1)^\alpha_e$ case $\alpha = \beta^* + 1$ and β^* even.)

“$\mathrm{cf}(\alpha) = \aleph_0$.” Given $\{r_i : i < \omega_1\} \subseteq R_p$, for $i < \omega_1$ let

$$F_i \stackrel{\text{def}}{=} \{\beta \in \mathrm{Dom}(p) : \neg(r_i \restriction \beta \Vdash \text{“}r_i(\beta) = p(\beta)\text{”})\}.$$

Without loss of generality, as $F_i \in [\mathrm{Dom}(p) \cap EVEN]^{<\aleph_0}$, we have, for all i, $F_i = F^*$, and now the conclusion follows by the induction hypothesis. $\blacksquare_{4.15}$

CLAIM 4.16: *Suppose that $\alpha \leq \omega_2$ and $q, r \in P'_\alpha$ are such that $p \leq_{\mathrm{apr}} r$ and $p \leq_{\mathrm{pr}} q$. Let us define $r + q$ by letting* $\mathrm{Dom}(r+q) = \mathrm{Dom}(q)$ *and, for* $\beta \in \mathrm{Dom}(r+q)$,

$$(r+q)(\beta) \stackrel{\text{def}}{=} \begin{cases} r(\beta) & \textit{if } \beta \in EVEN \in \mathrm{Dom}(r), \\ q(\beta) & \textit{otherwise.} \end{cases}$$

Then $r + q \in R_q$ *and* $r + q \geq_{\mathrm{pr}} r$.

Proof of the Claim: The proof is by induction on α, for all conditions in P'_α simultaneously. The eventful case of the induction is
“$\alpha = \beta + 1$, β even.”

We need to prove that $(r+q)(\beta)$ is simple above $(r+q) \restriction \beta$.

CASE 1: $\beta \in \mathrm{Dom}(r)$. Let $\langle I_n : n < \omega \rangle$ exemplify that $r(\beta)$ is simple above $r \restriction \beta$. For $n < \omega$ let

$$J_n \stackrel{\text{def}}{=} \{s + [(r+q) \restriction \beta] : s \in I_n\}.$$

By the induction hypothesis we have that J_n is a countable subset of $R_{[(r+q)\restriction\beta]}$. We finish by noticing that J_n is predense in $R_{[(r+q)\restriction\beta]}$.

CASE 2: $\beta \notin \mathrm{Dom}(r)$. Let now $\langle I_n : n < \omega \rangle$ exemplify that $q(\beta)$ is simple above $q \restriction \beta$. Let, for $n < \omega$,

$$\underset{\sim}{K}_n \stackrel{\text{def}}{=} \{z \in R_{[(r+q)\restriction\beta]} : (\exists s \in I_n)[z \geq s]\},$$

and let J_n be countable predense $\subseteq K_n$ (exists by 4.15). It is easily seen that $\langle J_n : n < \omega \rangle$ exemplify that $(r+q)(\beta)$ is simple above $(r+q) \restriction \beta$. $\blacksquare_{4.16}$

NOTATION 4.17: Suppose that p and q are as in Claim 4.16, and $R \subseteq R_p$. Then $R + q \stackrel{\text{def}}{=} \{r + q : r \in R\}$.

5. Properness

CLAIM 5.1: *Given $\alpha \leq \omega_2$, the following holds.*

$(1)^\alpha$ *P'_α is a $\leq_{\mathrm{pr}}$-dense subset of P_α.*

$(2)^\alpha$ *Suppose that $N \prec (H(\chi), \in)$ is countable, $\{p, \alpha, \bar{Q}, \bar{Q}'\} \subseteq N$, where p is some element of P'_α. Further assume that $J \in N$ is a $\leq_{\mathrm{pr}}$-open $\leq_{\mathrm{pr}}$-dense above p subset of $\{q \in P'_\alpha : p \leq_{\mathrm{pr}} q\}$, and u is finite $\subseteq ODD \cap \mathrm{Dom}(p)$, while $\epsilon > 0$. In addition, suppose that for $\gamma \in u$ we have a P_γ-name $\underset{\sim}{\tau}_\gamma$ (not necessarily in N) such that $p \restriction \gamma \Vdash$ "$\underset{\sim}{\tau}_\gamma$ is finite $\subseteq \underset{\sim}{T}^\gamma_{N\cap\omega_1}$". Let $\bar{\underset{\sim}{\tau}} \stackrel{\text{def}}{=} \langle \underset{\sim}{\tau}_\gamma : \gamma \in u \rangle$.*

Then *there is $q \in P'_\alpha$ such that*

$$(*)^\alpha_{p,q,N,J,u,\epsilon,\bar{\underset{\sim}{\tau}}} \text{ meaning } \begin{cases} \text{(i) } q \geq_{\mathrm{pr}} p, \\ \text{(ii) } q \text{ is } (N, P'_\alpha)\text{-generic, moreover,} \\ \text{(ii)}^+ \ q \text{ is a limit of a } \leq_{\mathrm{pr}}\text{-increasing} \leq_{\mathrm{pr}}\text{-generic} \\ \text{sequence } \langle q_n : n < \omega \rangle \text{ such that for every } I \in N \text{ an} \\ \quad \text{open dense} \subseteq P'_\alpha \bigcup_{n<\omega}(I \cap R_{q_n} \cap N) \text{ is predense} \\ \quad \text{above } q, \text{ while } q_0 \geq_{\mathrm{pr}} p \text{ and each } q_n \in N. \\ \text{(iii) For all } \gamma \in u \text{ and } \underset{\sim}{x} \text{ with } q \restriction \gamma \Vdash \text{“}\underset{\sim}{x} \in \underset{\sim}{\tau}_\gamma\text{”}, \\ \quad q \restriction \gamma \Vdash \text{“}\underset{\sim}{f}^{q(\gamma)}(\underset{\sim}{x}) < \underset{\sim}{f}^{p(\gamma)}(\underset{\sim}{x} \restriction (\delta^*(p)+1)) + \epsilon\text{”}, \\ \text{(iv) } \delta^*(q) = N \cap \omega_1 \text{ and} \\ \text{(v) } q \in J. \end{cases}$$

NOTATION 5.2: Suppose that $(*)^\alpha_{p,q,N,J,u,\epsilon,\bar{\underset{\sim}{\tau}}}$ holds for some appropriate values of $\alpha, p, q, N, J, u, \epsilon, \bar{\underset{\sim}{\tau}}$, and that $\langle q_n : n < \omega \rangle$ is a sequence as in the definition of $(*)^\alpha_{p,q,N,J,u,\epsilon,\bar{\underset{\sim}{\tau}}}$. We say that $\langle q_n : n < \omega \rangle$ **exemplifies** that $(*)^\alpha_{p,q,N,J,u,\epsilon,\bar{\underset{\sim}{\tau}}}$ holds.

Proof of the Claim: The proof is by induction on α, $(1)^\alpha$ and $(2)^\alpha$ simultaneously. However, we shall formulate four additional statements to help us carry the induction. These statements are denoted by $(1)^\alpha_e$, $(1)^{+,\alpha}$, $(1)^\alpha_o$ and $(2)^{+,\alpha}$. We shall prove by induction on α that $(1)^\alpha_e$, $(1)^{+,\alpha}$, $(1)^\alpha_o$, and $(2)^{+,\alpha}$ hold. As $(2)^{+,\alpha}$ is clearly a strengthening of $(2)^\alpha$ and $(1)^{+,\alpha}$ of $(1)^\alpha$, this suffices.

Description of $(1)^\alpha_e$, $(1)^{+,\alpha}$, $(1)^\alpha_o$ *and* $(2)^{+,\alpha}$.

$(1)^\alpha_e$ Assume that $\alpha = \beta+1$ and β is even, while $p \in P_\alpha$ is such that $p \restriction \beta \in P'_\beta$, $J \subseteq P'_\alpha$ is $\leq_{\rm pr}$-open and $\leq_{\rm pr}$-dense above p. Further assume that $N \prec (H(\chi), \in)$ is countable and $\{p, \beta, \bar{Q}, \bar{Q}', J\} \subseteq N$. Suppose $(*)^\beta_{p\restriction\beta, r, N, J\restriction\beta, u, \epsilon, \underset{\sim}{\bar{\tau}}}$ for some appropriate $u, \underset{\sim}{\bar{\tau}}$ and ϵ.

Then $q \overset{\rm def}{=} r \cup \{(\beta, p(\beta))\} \in P'_\alpha$ and $q \geq_{\rm pr} p$. If $p \in P'_\alpha$, then $(*)^\alpha_{p,q,N,J,u,\epsilon,\underset{\sim}{\bar{\tau}}}$.

$(1)^{+,\alpha}$ Suppose that $p \in P_\alpha$, $\beta \leq \alpha$ and $r \in P'_\beta$ are such that for some $p' \in P'_\beta$ with $p' \geq_{\rm pr} p \restriction \beta$ and some appropriate $N, J, \epsilon, \underset{\sim}{\bar{\tau}}$ we have $(*)^\beta_{p',r,N,J,u,\epsilon,\underset{\sim}{\bar{\tau}}}$. Then there is $q \in P'_\alpha$ such that $q \restriction \beta = r$ and $q \geq_{\rm pr} p$.

$(1)^\alpha_o$ Suppose that $\alpha = \beta + 1$ and β is odd. Given $N \prec (H(\chi), \in)$ countable such that $\alpha, \bar{Q}, \bar{Q}' \in N$ we let J, u, ϵ and $\underset{\sim}{\bar{\tau}}$ be as in the assumptions of $(2)^\alpha$. Let $\delta \overset{\rm def}{=} N \cap \omega_1$. Let $\{\underset{\sim}{I}_n\colon n < \omega\}$ enumerate all P_β-names of open dense subsets of $\underset{\sim}{Q}_\beta$ which are elements of N, each appearing infinitely often. Further assume that $\langle r_n\colon n < \omega\rangle$ exemplifies that $(*)^\beta_{p\restriction\beta, r, N, J\restriction\beta, u\cap\beta, \epsilon, \underset{\sim}{\bar{\tau}}\restriction\beta}$ holds.

Now assume that $\langle p_n\colon n < \omega\rangle$ is a sequence of conditions in P'_α with the following properties:

(a) $p_n \restriction \beta = r_n$ and $p_n \in N$.

(b) $p \leq_{\rm pr} p_n \leq_{\rm pr} p_{n+1}$.

(c) There is a series $\Sigma_{n<\omega}\epsilon_n$ with $\Sigma_{n<\omega}\epsilon_n < \epsilon$, such that for each $n < \omega$ the following statement $\oplus$ is forced by r:

$$\oplus \overset{\rm def}{=} \begin{cases} \text{“}\underset{\sim}{f}^{p_{n+1}(\beta)}(\underset{\sim}{x}) < \underset{\sim}{f}^{p_n(\beta)}(\underset{\sim}{x} \restriction (\delta^*(p_n)+1)) + \epsilon_n\text{”, for all } \underset{\sim}{x} \text{ with} \\ r_{n+1} \Vdash \text{“}\underset{\sim}{x} \in \underset{\sim}{w}^{\underset{\sim}{T}^\beta}_{\underset{\sim}{g}_{\beta-1}(n)}{}^{\delta^*(p_{n+1})} \cup \{y \restriction (\delta^*(p_{n+1}+1)) : y \in \underset{\sim}{T}_\beta\}\text{”}. \end{cases}$$

(e) $r \Vdash_{P_\beta}$ “$p_{n+1}(\beta) \in \underset{\sim}{I}_n$”.

Then the following defines a condition q in P'_α:

We let ${\rm Dom}(q) = {\rm Dom}(r) \cup \{\beta\}$ and $q \restriction \beta = r$. Further let $\underset{\sim}{f}^{q(\beta)}(\underset{\sim}{x})$ be

$$\underset{\sim}{f}^{p_n(\beta)}(\underset{\sim}{x})$$

if, for some j with $r_n \Vdash$ "$j \in \underset{\sim}{C}^{p_n(\beta)}$", we have

$$r_n \Vdash \text{“}\underset{\sim}{x} \in \underset{\sim}{w}^{\underset{\sim}{T}^{\beta}_j}_{\underset{\sim}{g}_{\beta-1}(n)} \cup \{y \restriction (\delta^*(p_n)+1) : y \in \underset{\sim}{T}_{\beta}\}\text{”}$$

and let it be

$$\Sigma_{n<\omega} \underset{\sim}{f}^{p_n(\beta)}(\underset{\sim}{x} \restriction (\delta^*(p_n)+1))$$

if $r \Vdash$ "$\underset{\sim}{x} \in \underset{\sim}{T}^{\beta}_{\delta}$".

We let $\underset{\sim}{C}^{q(\beta)} \stackrel{\text{def}}{=} (\bigcup_{n<\omega} \underset{\sim}{C}^{p_n(\beta)}) \cup \{\delta\}$. Let $\underset{\sim}{\Psi}^{q(\beta)} \stackrel{\text{def}}{=} \bigcup_{n<\omega} \underset{\sim}{\Psi}^{p_n(\beta)}$.

Moreover,

$$(*)^{\alpha}_{p,q,N,u\cup\{\beta\},J,\epsilon,\bar{\underset{\sim}{T}}}.$$

$(2)^{+,\alpha}$ For every $\beta \leq \alpha$ and $p, N, J, u, \epsilon, \bar{\underset{\sim}{T}}$ as in the hypothesis of $(2)^{\alpha}$, and $r \in P'_{\beta}$ such that $(*)^{\beta}_{p\restriction\beta,r,N,J\restriction\beta,u\cap\beta,\epsilon,\bar{\underset{\sim}{T}}\restriction\beta}$, there is $q \in P'_{\alpha}$ such that $(*)^{\alpha}_{p,q,N,J,u,\epsilon,\bar{\underset{\sim}{T}}}$ and $q \restriction \beta = r$.

Proof of $(1)^{\alpha}_{e}$, $(1)^{+,\alpha}$, $(1)^{\alpha}_{o}$, *and* $(2)^{+,\alpha}$:
"$\alpha = 0$." Trivial.
"$\alpha = \beta^* + 1$ and β^* is even."

$(1)^{\alpha}_{e}$ Hence $\beta = \beta^*$. It is easily seen that $q \in P_{\alpha}$ and $q \geq_{\text{pr}} p$. By the choice of r, in order to see that $q \in P'_{\alpha}$ we only need to check that $p(\beta)$ is simple above $q \restriction \beta$. Given $n < \omega$, let

$$I \stackrel{\text{def}}{=} \{s \in P_{\beta}\colon s \text{ determines } p(\beta) \text{ to degree } n\}.$$

Hence $I \subseteq P_{\beta}$ is open dense above $p \restriction \beta$, and certainly $I \in N$. Let $I' \stackrel{\text{def}}{=} I \cap P'_{\beta}$. By the induction hypothesis $(1)^{\beta}$, we have that I' is an open dense subset of P'_{β}. So, by the choice of r as a limit of a purely increasing sequence $\langle r_m : m < \omega\rangle$ (see $(\text{ii})^{+}$) we have that

$$I_n \stackrel{\text{def}}{=} \bigcup_{m<\omega} (I' \cap (R_{r_m} + r) \cap N)$$

predense above. Certainly I_n is countable and consists of conditions which are in $R_{(q\restriction\beta)}$, so I_n is as required. This shows that $q \in P'_{\alpha}$.

Suppose that $p \in P'_{\alpha}$. As we have $\Vdash_{P_{\beta}}$ "$\underset{\sim}{Q}_{\beta}$ is *ccc*", it follows by the usual arguments that q is (N, P_{α})-generic. As we have just proved that P'_{α} is $\leq_{\text{pr}}$-dense $\subseteq P_{\alpha}$, by the choice of $\langle r_n\colon n < \omega\rangle$ we can find a subsequence $\langle r_{n_k}\colon k < \omega\rangle$

such that choosing $q_k \stackrel{\rm def}{=} r_{n_k} \cup \{(\beta, p(\beta))\}$ we'll have shown that (ii)$^+$ from the definition of $(*)^{\alpha}_{p,q,N,J,u,\epsilon,\underset{\sim}{\mathcal{T}}}$ holds. If $u \subseteq ODD \cap \alpha$, then in fact $u \subseteq \beta$, so (iii) holds as well, by the choice of r. It is also easily seen that (iv) and (v) hold, noticing that $J \restriction \beta$ is $\leq_{\rm pr}$-dense and $\leq_{\rm pr}$-open above $p \restriction \beta$.

$(1)^{+,\alpha}$ Given $p \in P_\alpha$, without loss of generality, $\beta = \beta^*$. Now apply $(1)^{\alpha}_e$ to $r \cup \{(\beta^*, p(\beta^*))\}$.

$(1)^{\alpha}_o$ Does not apply.

$(2)^{+,\alpha}$ Without loss of generality, $\beta = \beta^*$. We let $q \stackrel{\rm def}{=} r \cup \{(\beta, p(\beta))\}$. By $(1)^{\alpha}_e$ it follows that $(*)^{\alpha}_{p,q,N,J,u,\epsilon,\underset{\sim}{\mathcal{T}}}$ holds.

"$\alpha = \beta^* + 1$ and β^* is odd."

$(1)^{\alpha}_e$ Does not apply.

$(1)^{+,\alpha}$ Follows by the induction hypothesis $(1)^{+,\beta^*}$.

$(1)^{\alpha}_o$ Hence $\beta^* = \beta$. This is like the proof of Claim 2.8, but we also get to use Claim 2.11. We first show that $\Vdash_{P_\beta}$ "$q(\beta) \in \underset{\sim}{Q}_\beta$". It is easily seen that

$$\Vdash_{P_\beta} \text{“}\underset{\sim}{C}^{q(\beta)} \text{ is a closed subset of } \delta + 1 \text{ with the last element } \delta\text{”}.$$

It is also easy to see, by the choice of $\langle p_n \colon n < \omega\rangle$, that P_β forces that $\underset{\sim}{\Psi}^{q(\beta)}$ is a countable set of promises, and that $\underset{\sim}{\Gamma} \in \underset{\sim}{\Psi}^{q(\beta)} \Longrightarrow \delta \in \underset{\sim}{C}(\underset{\sim}{\Gamma})$ (because the promises are in N), and $\underset{\sim}{C}(\underset{\sim}{\Gamma}) \supseteq \underset{\sim}{C}^{q(\beta)} \setminus \min(\underset{\sim}{C}(\underset{\sim}{\Gamma}))$. We have to check that P_β forces $\underset{\sim}{f}^{q(\beta)}$ to be a well defined function.

All information in the next three paragraphs is either true or forced by r to be true, and which one is the case is clear from the context:

For $\underset{\sim}{x} \in \mathrm{Dom}(\underset{\sim}{f}^{q(\beta)})$ for which $\underset{\sim}{f}^{q(\beta)}(\underset{\sim}{x})$ is defined by the first clause of its definition, the fact that $\underset{\sim}{f}^{q(\beta)}(\underset{\sim}{x})$ is well defined follows from the fact that p_n are increasing. For those $\underset{\sim}{x} \in \mathrm{Dom}(\underset{\sim}{f}^{q(\beta)})$ for which $\underset{\sim}{f}^{q(\beta)}(\underset{\sim}{x})$ is defined by the second clause of the definition, we have $r \Vdash$ "$\underset{\sim}{x} \in \underset{\sim}{\mathcal{T}}^{\beta}_{\delta}$". Hence $\underset{\sim}{x}$ is a $P_{\beta-1}$-name (this is where we use the fact that $\underset{\sim}{\mathcal{T}}^{\beta}$ is a $P_{\beta-1}$-name). We define a $P_{\beta-1}$-name $\underset{\sim}{h}$ of a function from ω to ω by $\underset{\sim}{h}(n) = m$ iff $\underset{\sim}{x} \restriction (\delta^*(p_n) + 1)$ is the m-th element of the increasing enumeration of $\underset{\sim}{\mathcal{T}}^{\beta}_{\delta^*(p_n)}$, as forced by $r \restriction (\beta - 1)$.

By the definition of $\underset{\sim}{g}_{\beta-1}$ we have that for all but finitely many n, it is forced by r that $\underset{\sim}{h}(n) < \underset{\sim}{g}_{\beta-1}(n)$. Hence for all but finitely many n we have that $\oplus$

from (c) in $(1)^\alpha_o$ holds for $\underset{\sim}{x}$ in question. Hence $\underset{\sim}{f}^{q(\beta)}(\underset{\sim}{x})$ is well defined.

Now it is also obvious that $\underset{\sim}{f}^{q(\beta)}$ is forced to be a partial monotonically increasing function into $\mathbb{Q}$. We can also see that the domain of $\underset{\sim}{f}^{q(\beta)}$ is forced to be $\bigcup_{i \in \underset{\sim}{C}^{q(\beta)}} \underset{\sim}{T}^{\beta}_i$, as this follows by the fact that $\Vdash_{P_{\beta-1}}$ "$\underset{\sim}{g}_{\beta-1}$ diverges to ∞."

The rest is easy to check.

$(2)^{+,\alpha}$ By the induction hypothesis, without loss of generality we have $\beta^* = \beta$ and $u = \{\beta^*\}$. For $n < \omega$ let $\epsilon_n \stackrel{\text{def}}{=} \epsilon/2^{n+2}$.

Let $\delta \stackrel{\text{def}}{=} N \cap \omega_1$. By Fact 2.7, we can find a sequence $\langle p_n : n < \omega \rangle$ which satisfies (a)–(e) in the statement of $(1)^\alpha_o$, where we have chosen $\langle r_n : n < \omega \rangle$ to exemplify $(*)^{\beta}_{p \restriction \beta, r, N, J \restriction \beta, u \cap \beta, \epsilon, \underset{\sim}{\bar{\tau}} \restriction \beta}$.

"α a limit ordinal." Both $(1)^\alpha_e$ and $(1)^\alpha_o$ are vacuously true. The following proof proves both $(1)^{+,\alpha}$ and $(2)^{+,\alpha}$. Assume $p^- \in P_\alpha$ and $\beta \leq \alpha$.

Case 1: $\mathrm{cf}(\alpha) = \aleph_0$. Let $\langle \alpha_n : n < \omega \rangle$ be a sequence in N which is increasing and cofinal in α, with $\alpha_0 = \beta$. Let $p \stackrel{\text{def}}{=} p_0 \geq_{\text{pr}} p^- \restriction \beta$ be such that $p_0 \in P'_\beta$. Without loss of generality, $p_0 \in N$. Let $\langle u_n : n < \omega \rangle$ be an increasing sequence of finite subsets of $ODD \cap N \cap \omega_2$, with $\bigcup_{n<\omega} u_n = N \cap ODD \cap \omega_2$. Let $\delta \stackrel{\text{def}}{=} N \cap \omega_1$. Let $\langle \delta_n : n < \omega \rangle$ be an increasing sequence of ordinals, cofinal in δ, and such that $\delta_0 = \delta^*(p)$. We are assuming that the assumptions of $(2)^{+,\alpha}$ hold.

By induction on $n < \omega$ we shall construct two sequences $\langle q_n : n < \omega \rangle$ and $\langle p_n : n < \omega \rangle$ such that:

(A) $p_0 = p$ and $q_0 = r$.

(B) $p_n \in P'_\alpha \cap N$ and $q_n \in P'_{\alpha_n}$.

(C) $\delta^*(p_n) \geq \delta_n$.

(D) $(*)^{\alpha_n}_{p_n \restriction q_n, q_n, N, J \restriction \alpha_n, u_n \cap \alpha_n, \epsilon, \underset{\sim}{\bar{\tau}} \restriction \alpha_n}$.

(E) $p_{n+1} \geq_{\text{pr}} p_n$.

(F) $q_{n+1} \restriction \alpha_n = q_n$ and $p_{n+1} \geq_{\text{pr}} p^- \restriction \alpha_{n+1}$ and $p_{n+1} \restriction \alpha_n = q_n$.

The induction goes through without problems. We now take $q = \bigcup_{n<\omega} q_n$.

Case 2: $\mathrm{cf}(\alpha) = \aleph_1$. The conclusion follows by the induction hypothesis. $\blacksquare_{5.1}$

Remark 5.3: Claim 5.1 in particular implies that P is a proper forcing notion.

Claim 5.4: *(1) For all $\alpha < \omega_2$ we have*

(i) $\emptyset \Vdash_{P_\alpha}$ "$\underset{\sim}{Q}_\alpha$ *has $\aleph_2$-pic**".

(ii) $\emptyset \Vdash_{P_\alpha}$ "$2^{\aleph_0} = \aleph_1$".

(iii) $P''_\alpha \stackrel{\text{def}}{=} \{p \in P'_\alpha\colon (\forall i \in \mathrm{Dom}(p))\, [p(i)$ *is a name from* $H_{<\aleph_1}(\mathrm{Ord})]\}$ *is dense in* P'_α.

(2) P *has* $\aleph_2 - cc$.

Proof of the Claim: The proof uses Fact 2.6 and is like the corresponding proof for countable support iterations, [Sh -f VIII, §2], which we quoted as Fact 2.5. Of course, notice that *ccc* trivially implies $\aleph_2$-pic*. $\blacksquare_{5.4}$

LEMMA 5.5: *It is possible to arrange the bookkeeping so that*

$$V^P \models \text{"there are no Suslin trees (in fact, all Aronszajn trees are special)"}.$$

Proof of the Lemma: This is standard, by $V \models$ "$2^{\aleph_1} = \aleph_2$" and Fact 2.6. $\blacksquare_{5.5}$

6. Sweetness revisited

Notation 6.1: Suppose that $\alpha \leq \beta \leq \omega_2$ and $p, q \in P'_\beta$.

(0) Suppose $p \leq q$. We write $p(\alpha) \neq q(\alpha)$ iff $\neg(q \restriction \alpha \Vdash$ "$q(\alpha) = p(\alpha)$").

(1) $p \leq^+ q$ iff $p \leq q$ and, for all α even with $p(\alpha) \neq q(\alpha)$, we have that $\underset{\sim}{t}^{q(\alpha)}$ is an object $t^{q(\alpha)}$, not just a name.

(2) $p \leq^+_{\mathrm{apr}} q$ iff $[p \leq_{\mathrm{apr}} q$ and $p \leq^+ q]$.

(3) $\geq^+$ and $\geq^+_{\mathrm{apr}}$ are defined in the obvious manner.

(4) $R^+_p \stackrel{\text{def}}{=} \{r \in R_p\colon r \geq^+_{\mathrm{apr}} p\}$.

Each R^+_p will be a sweetness model.

CLAIM 6.2: *Suppose that* $\alpha \leq \omega_2$ *and* $p \leq_{\mathrm{apr}} q \in P'_\alpha$. Then *for some* $q^+ \in P'_\alpha$ *we have* $p, q \leq^+_{\mathrm{apr}} q^+$.

Proof of the Claim: By induction on α. The only eventful case of the induction is the case when $\alpha = \beta + 1$ for some β even. As $p(\beta)$ is simple above $p \restriction \beta$ and $q \restriction \beta \in R_{p\restriction\beta}$, we can find $z \in R_{p\restriction\beta}$ with $z \geq_{\mathrm{apr}} q$ and such that z decides the value of $\underset{\sim}{t}^{p(\beta)}$. By the induction hypothesis we find $z^+ \in R^+_z$ such that $z^+ \geq_{\mathrm{apr}} p \restriction \beta, q \restriction \beta, z$. We define t by letting

$$t \stackrel{\text{def}}{=} \{\eta\colon z^+ \Vdash \text{"}\eta \in \underset{\sim}{t}^{p(\beta)}\text{"}\},$$

and let $q^+ \stackrel{\text{def}}{=} z^+ \cup \{(\beta, (t, \underset{\sim}{T}^{p(\beta)}))\}$. $\blacksquare_{6.2}$

Definition 6.3: Suppose that $\hat{\alpha} \le \omega_2$ and $p \in P'_{\hat{\alpha}}$. We define:

(1) For $r \in R_p$ we let $\mathrm{Dom}^*_p(r) \stackrel{\mathrm{def}}{=} \{\beta \in \mathrm{Dom}(p) : r(\beta) \neq p(\beta)\}$.

$\mathrm{Dom}^*_p(r)$ is the domain of r in R_p. Note that $\mathrm{Dom}^*\, p(r) \subseteq EVEN$.

(2) A sequence $\bar{x}$ is called **an assignment for** p if for some $\alpha > \sup(\mathrm{Dom}(p))$, which we denote by $\alpha(\bar{x})$, we have

$$\bar{x} = \langle\langle A^\gamma_m : m < \omega\rangle : \gamma \in \mathrm{Dom}(p) \cup \{\alpha\}\rangle$$

and each A^γ_m is a directed subset of $R^+_{p\restriction\gamma}$, while $\bigcup_{m<\omega} A^\gamma_m$ is dense in $R^+_{p\restriction\gamma}$; or if $\bar{x}$ has an initial segment with domain $(\mathrm{Dom}(p)\cup\{\alpha'\})$ which has the just mentioned properties.

We use the notation ${}^{\bar{x}}A^\gamma_m$ to denote $\bar{x}(\gamma, m)$.

The intended meaning of an assignment is an enumeration of equivalence classes of $R^+_{p\restriction\gamma}$ for γ in $\mathrm{Dom}(\bar{x})$.

(3) For $a \in [\omega_2]^{\le\aleph_0}$, we define

$$\mathrm{FA}_a \stackrel{\mathrm{def}}{=} \{\langle(\beta_j, t_j) \colon j < j^*\rangle : j^* < \omega \;\&\; \beta_j \in a \cap EVEN \;\&\; \beta_j \text{ are increasing } \&\; t_j \text{ is a finite subtree of } {}^{<\omega}2\}.$$

The intended meaning of FA_a is to be a formal E_0-equivalence class.

(4) For $y \in \mathrm{FA}_{\mathrm{Dom}(p)}$, we let

$$A^y_p \stackrel{\mathrm{def}}{=} \{r \in R^+_p \colon \mathrm{Dom}^*_p(r) = \{\beta \colon (\exists t)((\beta, t) \in \mathrm{Rang}(y))\} \;\&\; (\beta, t) \in \mathrm{Rang}(y) \Longrightarrow t^{r(\beta)} = t\}.$$

(5) $\bar{y}$ is **a formal 0-canonical assignment for** p if, for some $\alpha > \sup(\mathrm{Dom}(p))$, which we denote by $\alpha(\bar{y})$, we have

$$\bar{y} = \langle \bar{y}^\gamma = \langle y^\gamma_m \colon m < \omega\rangle : \gamma \in \mathrm{Dom}(p) \cup \{\alpha\}\rangle$$

and $\{y^\gamma_m : m < \omega\}$ is a list, possibly with repetitions, of $\mathrm{FA}_{\mathrm{Dom}(p\restriction\gamma)}$, for $\gamma \in \mathrm{Dom}(p) \cup \{\alpha(\bar{y})\}$; or if $\bar{y}$ has an initial segment of domain $(\mathrm{Dom}(p) \cup \{\alpha'\})$ which has the just mentioned properties.

A formal 0-canonical assignment gives a list of formal E_0-equivalence classes. The main definition of this section, Definition 6.5, will deal with formal E_n-equivalence classes.

(6) An assignment $\bar{x}$ is **a 0-canonical assignment for** p if for some formal 0-canonical assignment $\bar{y}$ for p, we have ${}^{\bar{x}}A^\gamma_m \subseteq A^{y^\gamma_m}_p$, for all $\gamma \in \mathrm{Dom}(p) \cup \{\alpha(\bar{y})\}$ and $m < \omega$. We let, without loss of generality, $\alpha(\bar{x}) \stackrel{\mathrm{def}}{=} \alpha(\bar{y})$.

CLAIM 6.4: *Suppose that $p \in P'_{\hat\alpha}$ and $\beta < \hat\alpha$. Suppose that $\bar y$ is a formal 0-canonical assignment (assignment, 0-canonical assignment) for p.* Then $\bar y \restriction [\mathrm{Dom}(p \restriction \beta) \cup \{\alpha(\bar y)\}]$ *is a formal 0-canonical assignment (assignment, 0-canonical assignment) for $p \restriction \beta$.*

Proof of the Claim: Check, looking at (2), (5) and (6) of Definition 6.3. $\blacksquare_{6.4}$

We abuse the notation, and in the situation like that of Claim 6.4, we use $\bar y$ for $\bar y \restriction [\mathrm{Dom}(p \restriction \beta) \cup \{\alpha(\bar y)\}]$.

Definition 6.5: By simultaneous induction on $\hat\alpha \le \omega_2$ we define the following notions (a)–(d) and prove Lemma 6.6:

(a) For $a \in [\hat\alpha]^{\le\aleph_0}$, sets $\mathrm{FE}_n(a)$ for $n < \omega$. The elements of $\mathrm{FE}_n(a)$ are called **formal equivalence classes**.

These are intended as formal E_n-equivalence classes.

(b) For $a \in [\hat\alpha]^{\le\aleph_0}$, we define
  (1) For $b \trianglelefteq a \in [\hat\alpha]^{\le\aleph_0}$, a function $F_{b,a}\colon \bigcup_{n<\omega} \mathrm{FE}_n(a) \to \bigcup_{n<\omega} \mathrm{FE}_n(b)$.

F is intended as a restriction to a smaller domain.

  (2) Functions $\mathrm{Proj}^{n_2}_{n_1}(a)\colon \mathrm{FE}_{n_2}(a) \to \mathrm{FE}_{n_1}(a)$, for $n_1 \le n_2 < \omega$.

(c) For $a \in [\hat\alpha]^{\le\aleph_0}$, we define functions His_a and Base_a by defining
  (i) $\mathrm{His}_a(\hat\Upsilon)$ for $\hat\Upsilon \in \bigcup_{n<\omega} \mathrm{FE}_n(a)$.

His stands for history.

  (ii) $\mathrm{Base}_a(\hat\Upsilon)$ for $\hat\Upsilon \in \bigcup_{n<\omega} \mathrm{FE}_n(a)$.

(d) For $p \in P'_{\hat\alpha}$ and an assignment $\bar x$ for p we define when $\bar x$ is a **canonical assignment for** p.

(e) (I) For $p \in P_{\hat\alpha}$ and $n < \omega$ we define $\mathrm{type}^{p,n}_{\bar x} : R^+_p \to \mathrm{FE}_n(a)$, for $a \in [\hat\alpha]^{\le\aleph_0}$. Here $\bar x$ is a canonical assignment for p.
  (II) For $p, \bar x$ as in (I), we define an equivalence relation $E^{p,n}_{\bar x}$ on R^+_p.

LEMMA 6.6: *Suppose that $p \in P'_{\hat\alpha}$, and $\bar x$ is a canonical assignment for p.* Then

$$(1)^{\hat\alpha} \qquad \mathfrak{B}_{p,\bar x} \stackrel{\mathrm{def}}{=} \Big(R^+_p, \bigcup_{m<\omega}^{\bar x} A^{\alpha(\bar x)}_m, E^{p,n}_{\bar x}\Big)_{n<\omega}$$

is a sweetness model.

$(2)^{\hat\alpha,\beta}$ *Suppose that $\beta \le \hat\alpha$.* Then $\mathfrak{B}_{p\restriction\beta,\bar x} < \mathfrak{B}_{p,\bar x}$.

$(3)^{\hat\alpha}$ *For $b \trianglelefteq a \in [\hat\alpha]^{\le\aleph_0}$, we have that $F_{b,a}$ is a totally defined function.*

We proceed to give the inductive definition and proof.

"$\hat\alpha = 0$." In this case $p = \emptyset$ and $a = \emptyset$. We let

(a) $\mathrm{FE}_n(\emptyset) \stackrel{\mathrm{def}}{=} \{\langle n, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle\}$ for $n < \omega$.

(b) (1) $F_{\emptyset,\emptyset}$ is the identity.

(2) $\mathrm{Proj}^{n_2}_{n_1}(\emptyset) : \mathrm{FE}_{n_2}(\emptyset) \to \mathrm{FE}_{n_1}(\emptyset)$ is given by

$$\mathrm{Proj}^{n_2}_{n_1}(\emptyset)(\langle n_2, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle) \stackrel{\mathrm{def}}{=} \langle n_1, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle,$$

for $n_1 \leq n_2 < \omega$.

(c) (i) $\mathrm{His}_\emptyset(\langle n, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle) \stackrel{\mathrm{def}}{=} \{\langle n_1, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle : n_1 \leq n\}$ for $n < \omega$.

(ii) $\mathrm{Base}_\emptyset(\langle n, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle) \stackrel{\mathrm{def}}{=} \emptyset$.

(d) Any 0-canonical assignment for $\emptyset$ is a canonical assignment.

(e) (I) For $n < \omega$ we let $\mathrm{type}^{\emptyset,n}_\emptyset(\emptyset) \stackrel{\mathrm{def}}{=} \langle n, 0, 0, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\rangle$.

(II) For $n < \omega$, we let $\emptyset E^{\emptyset,n}_\emptyset \emptyset$.

Proof of the Lemma: [6.6, case $\hat\alpha = 0$]. Trivial. $\blacksquare_{6.6,\hat\alpha=0}$

"$\hat\alpha = \hat\beta + 1$." We first consider (a), (b) and (c) above. Fix $a \in [\hat\alpha]^{\leq\aleph_0}$.

CASE 1: [$\hat\beta$ is odd] or [$\hat\beta$ is even & $\hat\beta \notin a$].

(a) For $n < \omega$, let $\mathrm{FE}_n(a) \stackrel{\mathrm{def}}{=} \mathrm{FE}_n(a \cap \hat\beta)$.

(b) (1) For $b \trianglelefteq a \in [\hat\alpha]^{\leq\omega_2}$ and $n < \omega$, we let $F_{b,a} \stackrel{\mathrm{def}}{=} F_{b\cap\hat\beta, a\cap\hat\beta}$.

(2) For $n_1 \leq n_2 < \omega$, let $\mathrm{Proj}^{n_2}_{n_1}(a) \stackrel{\mathrm{def}}{=} \mathrm{Proj}^{n_2}_{n_1}(a \cap \hat\beta)$.

(c) (i) $\mathrm{His}_a \stackrel{\mathrm{def}}{=} \mathrm{His}_{a\cap\hat\beta}$.

(ii) $\mathrm{Base}_a \stackrel{\mathrm{def}}{=} \mathrm{Base}_{a\cap\hat\beta}$.

CASE 2 (main case): *$\hat\beta$ is even and $\hat\beta \in a$.*

(a) For $n < \omega$,

$$\mathrm{FE}_n(a) \stackrel{\mathrm{def}}{=} \mathrm{FE}_n(a \cap \hat\beta) \cup \{\langle n, 1, \hat\beta, \Upsilon, t, w, u, \bar k, \bar\varepsilon\rangle : (*) \text{ holds }\},$$

where for $(*)$ to hold it means that the following six items are satisfied:

1. $\Upsilon \in \mathrm{FE}_k(a\cap\hat\beta)$ for some $k \geq n$ (the equivalence class of the initial segment),
2. $(\exists \hat m)$ (t is a subtree of ${}^{<\omega}2$ of height $\hat m$),
3. $w \subseteq \{0, \ldots, n-1\}$ (the places where there is an extension in the corresponding equivalence class),
4. $u \subseteq \{(\eta, m): \eta \in {}^{<n}2 \text{ and } m \in w\}$ (for $m \in w$, the witness that the m-th equivalence class in the enumeration produces to show that $\underset{\sim}{\mathcal{T}}^{r(\hat\beta)}$ is nowhere dense),
5. $\bar k = \langle (k_m, \Upsilon_m): m \in w\rangle$ is such that

$$(\forall m \in w)\,[k_m < \omega \;\&\; \Upsilon_m \in \mathrm{FE}_{k_m}(a \cap \hat\beta)]$$

(the sequence of k's for the equivalence classes of the projections),

6. $\bar{\varepsilon}$ is an increasing finite sequence with $\mathrm{Rang}(\bar{\varepsilon}) \subseteq a \cap EVEN$

(the coordinates where the equivalence class lives).

We let, for $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a)$,

$$\hat{\Upsilon} \stackrel{\text{def}}{=} \langle n^{[\hat{\Upsilon}]}, o^{[\hat{\Upsilon}]}, \beta^{[\hat{\Upsilon}]}, \Upsilon^{[\hat{\Upsilon}]}, t^{[\hat{\Upsilon}]}, w^{[\hat{\Upsilon}]}, u^{[\hat{\Upsilon}]}, \bar{k}^{[\hat{\Upsilon}]}, \bar{\varepsilon}^{[\hat{\Upsilon}]} \rangle.$$

(b) (1) We define $F_{b,a}$ by cases:

SUBCASE 1: $a = b$. $F_{b,a}$ is the identity.

SUBCASE 2: *$b \neq a$ and $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a \cap \hat{\beta})$.* $F_{b,a}(\hat{\Upsilon}) = F_{b,a \setminus \{\hat{\beta}\}}(\hat{\Upsilon})$.

SUBCASE 3: *None of the Subcases 1 and 2 hold.* $F_{b,a}(\hat{\Upsilon}) = F_{b,a \setminus \{\hat{\beta}\}}(\Upsilon^{[\hat{\Upsilon}]})$.

(2) For $n_1 \leq n_2 < \omega$ we let $\left(\mathrm{Proj}^{n_2}_{n_1}(a)\right)(\Upsilon_2) = \Upsilon_1$ iff

SUBCASE 1: $\Upsilon_2 \in \mathrm{FE}_{n_2}(a \cap \hat{\beta})$ *and* $\Upsilon_1 = \left(\mathrm{Proj}^{n_2}_{n_1}(a \cap \hat{\beta})\right)(\Upsilon_2)$.

SUBCASE 2: new $\Upsilon_2 \notin \bigcup_{n<\omega} \mathrm{FE}_n(a \cap \hat{\beta})$ but $\Upsilon_2 \in \mathrm{FE}_{n_2}(a)$ and Υ_1 satisfies (α)–(η) below, if possible:

(α) $n^{[\Upsilon_1]} = n_1$, while $o^{[\Upsilon_1]} = 1$ and $\beta^{[\Upsilon_1]} = \hat{\beta}$,

(β) $\Upsilon^{[\Upsilon_1]} = (\mathrm{Proj}^{n_2}_{n_1}(a \cap \hat{\beta}))(F_{a\cap\hat{\beta},a}(\Upsilon_2))$,

(γ) $t^{[\Upsilon_1]} = t^{[\Upsilon_2]}$,

(δ) $w^{[\Upsilon_1]} = w^{[\Upsilon_2]} \cap n_1$,

(ε) $u^{[\Upsilon_1]} = u^{[\Upsilon_2]} \cap \{(\eta, m) : \eta \in {}^{<n_1}2 \ \&\ m \in w^{[\Upsilon_1]}\}$,

(ζ) $\bar{k}^{[\Upsilon_1]} = \bar{k}^{[\Upsilon_2]} \restriction w^{[\Upsilon_1]}$,

(η) $\bar{\varepsilon}^{[\Upsilon_1]} = \bar{\varepsilon}^{[\Upsilon_2]}$.

SUBCASE 3: If $F_{b,a}(\Upsilon)$ has not been defined by any of the two subcases above, we leave it undefined.

(c) (i) $\mathrm{His}_a(\hat{\Upsilon})$ is given by

$$\mathrm{His}_a(\hat{\Upsilon}) = \{\hat{\Upsilon}\} \cup \bigcup_{n_1 \leq n^{[\hat{\Upsilon}]}} \mathrm{His}_{a\setminus\{\hat{\beta}\}}(F_{a\setminus\{\hat{\beta}\},a}(\mathrm{Proj}^{n^{[\hat{\Upsilon}]}}_{n_1}(a))(\hat{\Upsilon}))$$
$$\cup \bigcup_{m \in w^{[\hat{\Upsilon}]}} \mathrm{His}_{a\setminus\{\hat{\beta}\}}((\mathrm{Proj}^{n^{[\hat{\Upsilon}]}}_{m}(a \setminus \{\hat{\beta}\}))(\Upsilon^{[\hat{\Upsilon}]}).$$

(ii) $\mathrm{Base}_a(\hat{\Upsilon}) = \{(\beta^{[\Upsilon]}, n_1) : \Upsilon \in \mathrm{His}_a(\hat{\Upsilon}) \ \&\ n_1 \leq n^{[\hat{\Upsilon}]}\}$.

We go on to define (d), (e) for the case $\hat{\alpha} = \hat{\beta} + 1$.

(d) Let $\bar{x}$ be a 0-canonical assignment for $p \in P'_{\hat{\alpha}}$.

SUBCASE 1: $\hat\beta \notin \mathrm{Dom}(p)$. $\bar x$ is a canonical assignment for p iff $\bar x$ is a canonical assignment for $p \restriction \hat\beta$.

SUBCASE 2: $\hat\beta \in \mathrm{Dom}(p)$. $\bar x$ is a canonical assignment for p if $\bar x$ is a canonical assignment for $p \restriction \hat\beta$ <u>and</u> $\langle {}^{\bar x}A^{\hat\beta}_m : m < \omega\rangle$ is an enumeration of all $E^{p\restriction\hat\beta,n}_{\bar x\restriction\hat\beta}$-equivalence classes for $n < \omega$.

(e) For $n < \omega$ and $\bar x$ a canonical assignment for p, we define the function $\mathrm{type}^{p,n}_{\bar x} : R^+_p \to \mathrm{FE}_n(a)$ by describing $\mathrm{type}^{p,n}_{\bar x}(r)$ for $r \in R^+_p$.

SUBCASE 1: $\hat\beta \notin \mathrm{Dom}^*_p(r)$ *or* $\hat\beta \notin \mathrm{Dom}(p)$. We let $\mathrm{type}^{p,n}_{\bar x}(r) \stackrel{\mathrm{def}}{=} \mathrm{type}^{p\restriction\hat\beta,n}_{\bar x}(r \restriction \hat\beta)$.

SUBCASE 2: $\hat\beta \in \mathrm{Dom}^*_p(r)$. We shall have $\mathrm{type}^{p,n}_{\bar x}(r) = \hat\Upsilon$ for some $\hat\Upsilon \in \mathrm{FE}_n(a)$. We define $\hat\Upsilon$ by defining its nine coordinates

$$\langle n^{[\hat\Upsilon]}, o^{[\hat\Upsilon]}, \beta^{[\hat\Upsilon]}, \Upsilon^{[\hat\Upsilon]}, t^{[\hat\Upsilon]}, w^{[\hat\Upsilon]}, u^{[\hat\Upsilon]}, \bar k^{[\hat\Upsilon]}, \bar\varepsilon^{[\hat\Upsilon]}\rangle.$$

We'll have $n^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} n$, $o^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} 1$, and $\beta^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} \hat\beta$. Furthermore, $t^{[\hat\Upsilon]} = t^{r(\hat\beta)}$. Arriving at the heart of the matter,

$$w^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} \{m < n : r \restriction \hat\beta \text{ has an extension in } {}^{\bar x}A^{\hat\beta}_m\},$$

while $u^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} \{(\eta, m)\colon \eta \in {}^{<n}2 \;\&\; m \in w^{[\hat\Upsilon]} \&$ for some $q \in {}^{\bar x}A^{\hat\beta}_m$ we have $q \Vdash$ "$\eta \notin \underset{\sim}{\mathcal{T}}^{r(\hat\beta)}$"$\}$, $\bar k^{[\hat\Upsilon]} = \langle\langle k_m(r \restriction \hat\beta), \mathrm{type}^{p\restriction\hat\beta,m}_{\bar x}(r \restriction \hat\beta)\rangle\colon m \in w^{[\hat\Upsilon]}\rangle$, where

$$k_m(r \restriction \hat\beta) \stackrel{\mathrm{def}}{=} \min\left\{k\colon \left(\forall q \in (r \restriction \hat\beta)/E^{p\restriction\hat\beta,k}_{\bar x}\right)(\exists q' \in {}^{\bar x}A^{\hat\beta}_m)\,(q' \geq q)\right\}.$$

The fact that such numbers $k_m(r \restriction \hat\beta)$ are well defined is a part of the induction hypothesis (see Definition 2.13). Let

$$k^{[\hat\Upsilon]} \stackrel{\mathrm{def}}{=} \mathrm{Max}(\{k_m(r \restriction \hat\beta)\colon m < n\} \cup \{n\}).$$

We'll have

$$\Upsilon^{[\hat\Upsilon]} = \mathrm{type}^{p\restriction\hat\beta,k^{[\hat\Upsilon]}}_{\bar x}(r \restriction \hat\beta).$$

Finally, $\bar\varepsilon^{[\hat\Upsilon]}$ is the increasing list of $\mathrm{Dom}^*_p(r)$.

To see that the definition is well posed, notice that $r \restriction \hat\beta \in R^+_{p\restriction\hat\beta}$.

(II) For $r', r'' \in R^+_p$, we let

$$r' E^{p,n}_{\bar x} r'' \text{ iff } \mathrm{type}^{p,n}_{\bar x}(r') = \mathrm{type}^{p,n}_{\bar x}(r'').$$

Proof of the Lemma: [6.6, case $\hat\alpha = \hat\beta + 1$]. Without loss of generality, $\beta = \hat\beta$. We prove $(2)^{\hat\alpha,\beta}$, and $(1)^{\hat\alpha}$ follows. By comparing with Definition 2.14 (which is [Sh 176, 7.6]), we can see that $\mathfrak{B}_{p,\bar{x}}$ is isomorphic to the canonical sweetness model on $R^+_{p\restriction\beta} * \underset{\sim}{UM}$ with respect to $\mathfrak{B}_{p\restriction\beta,\bar{x}}$. Notice that UM is a homogeneous forcing notion. The conclusion follows from the Composition Lemma 2.17 (which is [Sh 176, 7.6–7.9]).

$(3)^{\hat{\alpha}}$ Follows from $(2)^{\hat{\alpha},\beta}$. $\blacksquare_{6.6,\hat{\alpha}=\hat{\beta}+1}$

"$\hat{\alpha}$ is a limit ordinal."

(a) For $a \in [\hat{\alpha}]^{\leq\aleph_0}$, we consider two cases:

Case 1: $\sup(a) = \alpha < \hat{\alpha}$. $\mathrm{FE}_n(a)$ is already defined by the induction hypothesis.

Case 2: $\sup(a) = \hat{\alpha}$. We let $\mathrm{FE}_n(a) \stackrel{\text{def}}{=} \bigcup_{\alpha\in a} \mathrm{FE}_n(a \cap \alpha)$.

(b) We again consider two cases.

Case 1: $\sup(a) = \alpha < \hat{\alpha}$.

(1) For $b \trianglelefteq a$, we have already defined $F_{b,a}$.

(2) Functions $\mathrm{Proj}^{n_2}_{n_1}(a)$ are defined by the induction hypothesis, for $n_1 \leq n_2 < \omega$.

Case 2: $\sup(a) = \hat{\alpha}$.

(1) Subcase 1. $b = a$.

We define $F_{a,a}$ as the identity.

Subcase 2. $b \neq a$.

Suppose that $n < \omega$ and $\hat{\Upsilon} \in \mathrm{FE}_n(a)$. Let $\alpha < \hat{\alpha}$ be large enough such that $\hat{\Upsilon} \in \mathrm{FE}_n(a \cap \alpha)$ and $b \trianglelefteq (a \cap \alpha)$. We let $F_{b,a}(\hat{\Upsilon}) \stackrel{\text{def}}{=} F_{b,a\cap\alpha}(\hat{\Upsilon})$.

(2) For $n_1 \leq n_2 < \omega$ and $\hat{\Upsilon} \in \mathrm{FE}_{n_2}(a)$, we define

$$(\mathrm{Proj}^{n_2}_{n_1}(a))(\hat{\Upsilon}) \stackrel{\text{def}}{=} (\mathrm{Proj}^{n_2}_{n_1}(a \cap \alpha))(\hat{\Upsilon})$$

if $\hat{\Upsilon} \in \mathrm{FE}_{n_2}(a \cap \alpha)$.

(c) (i) For $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a)$, we define $\mathrm{His}_a(\hat{\Upsilon}) \stackrel{\text{def}}{=} \mathrm{His}_{a\cap\alpha}(\hat{\Upsilon})$ for any $\alpha < \hat{\alpha}$ such that $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a \cap \alpha)$.

(ii) For $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a)$, we let $\mathrm{Base}_a(\hat{\Upsilon}) \stackrel{\text{def}}{=} \mathrm{Base}_{a\cap\alpha}(\hat{\Upsilon})$ for α such that $\hat{\Upsilon} \in \bigcup_{n<\omega} \mathrm{FE}_n(a \cap \alpha)$.

(d) $\bar{x}$ is a canonical assignment for $p \in P'_{\hat{\alpha}}$ iff for all $\beta < \hat{\alpha}$ we have that $\bar{x}$ is a canonical assignment for $p \restriction \beta$.

(e) (I) Suppose that $n < \omega$. For $r \in R^+_p$ we let

$$\mathrm{type}^{p,n}_{\bar{x}}(r) \stackrel{\text{def}}{=} \mathrm{type}^{p\restriction\alpha,n}_{\bar{x}}(r \restriction \alpha)$$

for any $\alpha < \hat{\alpha}$ such that $\mathrm{Dom}^*_p(r) \subseteq \alpha$.

(II) For $n < \omega$ and $r', r'' \in R^+_p$, we let

$$r' E^{p,n}_{\bar{x}} r'' \text{ iff } (r' \restriction \alpha) E^{p\restriction\alpha,n}_{\bar{x}} (r'' \restriction \alpha)$$

for any $\alpha < \hat{\alpha}$ such that $\mathrm{Dom}^*_p(r') \cup \mathrm{Dom}^*_p(r'') \subseteq \alpha$.

As a part of the inductive definition in the case $\hat{\alpha}$ a limit ordinal, we prove the following

Observation 6.7: Objects in items (a)–(e) above are well defined.

Proof of the Observation: We have to check several spots where the definition in the case of $\hat{\alpha}$ limit might run into a contradiction. We start by (b) Case 2(1), Subcase 2. We assume that $\alpha_1 \leq \alpha_2 < \hat{\alpha}$, while $\hat{\Upsilon} \in \mathrm{FE}_n(a \cap \alpha_1) \cap \mathrm{FE}_n(a \cap \alpha_2)$, and $b \trianglelefteq a \cap \alpha_1$. We can prove by induction on $\alpha \in [\alpha_1, \alpha_2]$ that $F_{b,a\cap\alpha_1}(\hat{\Upsilon}) = F_{b,a\cap\alpha}(\hat{\Upsilon})$. Note the definition in the case that $\hat{\alpha}$ is a successor ordinal, item (b)(1), Subcase 2 of Case 2.

We move on to (b), Case 2(2). Suppose that $\alpha_1 \leq \alpha_2 < \hat{\alpha}$ and $\hat{\Upsilon} \in \mathrm{FE}_{n_2}(a \cap \alpha_1) \cap \mathrm{FE}_{n_2}(a \cap \alpha_2)$. We can prove by induction on $\alpha \in [\alpha_1, \alpha_2]$ that

$$(\mathrm{Proj}^{n_2}_{n_1}(a \cap \alpha_1))(\hat{\Upsilon}) = (\mathrm{Proj}^{n_2}_{n_1}(a \cap \alpha))(\hat{\Upsilon}).$$

Observe the way the definition is set up in Subcase 1 of Case 2, (b)(2) of the definition for the case of $\hat{\alpha}$ being a successor ordinal.

We go to item (c), part (i), which is proved similarly, observing the set up of the definition in the case of $\hat{\alpha}$ being a successor ordinal, Case 2, item (c) (i). Similarly for item (c), part (ii).

We still have to check items (d) and (e), which is done in a similar fashion. $\blacksquare_{6.7}$

Proof of the Lemma: [6.6, case $\hat{\alpha}$ a limit]. First suppose $\mathrm{cf}(\hat{\alpha}) = \aleph_0$. We prove $(2)^{\hat{\alpha},\beta}$ for a given $\beta \leq \hat{\alpha}$. Without loss of generality, $\beta < \hat{\alpha}$. Let $\langle \alpha_n : n < \omega \rangle$ be an increasing sequence of ordinals with $\alpha_0 = \beta$ and $\sup_{n<\omega} \alpha_n = \hat{\alpha}$. Considering $\mathfrak{B}_{p\restriction\alpha_n,\bar{x}}$ $(n < \omega)$, we finish by the induction hypothesis and Fact 2.18.

If $\mathrm{cf}(\hat{\alpha}) \geq \aleph_1$, the conclusion follows by the induction hypothesis. $\blacksquare_{6.6}$

This ends the inductive definition.

Definition 6.8: Suppose that $\alpha \leq \omega_2$ and $p \leq_{\mathrm{pr}} q \in P'_\alpha$.

By induction on α we define $(\mathrm{A})^\alpha$ and prove $(\mathrm{B})^\alpha$ below:

(A)$^\alpha$ Suppose that $\bar{x}$ is a canonical assignment for q with $\alpha(\bar{x}) \leq \alpha$. We define $\bar{x} : p$ by letting

$$\bar{x}\colon p' \stackrel{\text{def}}{=} \langle {}^{\bar{x}\colon p'}A^{\beta}_{m}\colon \beta \in [\mathrm{Dom}(\bar{x}) \cap (\mathrm{Dom}(p)] \cup \{\alpha(\bar{x})\})\rangle,$$

and for $\beta \in \mathrm{Dom}(\bar{x}\colon p)$, for the unique $n < \omega$ and $\Upsilon \in \mathrm{FE}_n(\mathrm{Dom}(p))$ such that

$${}^{\bar{x}}A^{\beta}_{m} = \{z \in R^{+}_{q\restriction\beta}\colon \mathrm{type}^{q\restriction\beta,n}_{\bar{x}}(z) = \Upsilon\},$$

we have

$${}^{\bar{x}\colon p}A^{\beta}_{m} \stackrel{\text{def}}{=} \{\check{z} \in R^{+}_{p\restriction\beta}\colon \mathrm{type}^{p\restriction\beta,n}_{\bar{x}\colon p}(\check{z}) = \Upsilon\}.$$

(B)$^\alpha$

Claim 6.9: *Suppose that $\bar{x}$ is a canonical assignment for q with $\alpha(\bar{x}) \leq \alpha$.* Then *$\bar{x} : p$ is a canonical assignment for p.*

Proof of the Claim: Check Definition 6.5. $\blacksquare_{6.9}$

7. More partial orders

Claim 7.1: *Suppose that $\alpha \leq \omega_2$, while $p \leq p^* \in P'_\alpha$ and $q_1, q_2 \in R_p$ are such that $q_1, q_2 \leq p^*$. Then there is $p^{**} \geq p^*$ and $q^* \in R_p$ such that $q_1, q_2 \leq_{\mathrm{apr}} q^*$ and $q^* \leq p^{**}$, and $p^{**} \in P_{\alpha'}$.*

Proof of the Claim: The proof is by induction on α. The eventful case of the induction is when $\alpha = \beta + 1$ for some even $\beta \in \mathrm{Dom}(p)$ such that

$$\neg(q_1 \restriction \beta \Vdash \text{“}q_1(\beta) = p(\beta)\text{” and } q_2 \restriction \beta \Vdash \text{“}q_2(\beta) = p(\beta)\text{”}).$$

We can find $p' \geq p^* \restriction \beta$ in P'_β which forces a value to all

$$\underset{\sim}{v}_0 \stackrel{\text{def}}{=} \underset{\sim}{t}^{\,p^*(\beta)}, \quad \underset{\sim}{v}_1 \stackrel{\text{def}}{=} \underset{\sim}{t}^{\,q_1(\beta)}, \quad \underset{\sim}{v}_2 \stackrel{\text{def}}{=} \underset{\sim}{t}^{\,q_2(\beta)}.$$

By the induction hypothesis, possibly extending p', there is $q' \in R_{p\restriction\beta}$ such that $q_1 \restriction \beta$, $q_2 \restriction \beta \leq_{\mathrm{apr}} q'$ and $q' \leq p'$. We know that there is a predense set J in $R_{p\restriction\beta}$ such that each condition in J forces all of the above values (here is where we use the notion of “simple above”). Possibly increasing p' we can assume that there are $r_0, r_1, r_2 \in R_{p\restriction\beta}$ forcing a value to $\underset{\sim}{v}_0, \underset{\sim}{v}_1, \underset{\sim}{v}_2$ respectively, and all below p'.

By the induction hypothesis, possibly extending p' again, there is $q \in R_{p\restriction\beta}$ which is above q' and r_0–r_2, and below p'. Let

$$q^* \stackrel{\text{def}}{=} \begin{array}{l} q \cup \{(\beta, (\{\eta : q \Vdash \text{“}\eta \in \underset{\sim}{t}^{\,q_1(\beta)} \cup \underset{\sim}{t}^{\,q_2(\beta)}\text{”}\}, \\ \underset{\sim}{\mathcal{T}}^{q_1(\beta)} \cup \underset{\sim}{\mathcal{T}}^{q_2(\beta)}))\}, \end{array}$$

and $p^{**} \stackrel{\text{def}}{=} p' \cup \{(\beta, q^*(\beta))\}$. We need to check that $q^*(\beta)$ is simple above q (so above p'), which follows as $q_l(\beta)$ is simple above $q_l \restriction \beta$ for $l = 1, 2$. $\blacksquare_{7.1}$

COROLLARY 7.2: *If α, p, q_1, q_2 are as in Claim 7.1, then q_1, q_2 are compatible in P'_α iff q_1, q_2 are compatible in R_p.*

Definition 7.3: Suppose that $\alpha \leq \omega_2$ and $u \subseteq \mathrm{Dom}(p)$. We define

(1) $GR_{p,u} \stackrel{\text{def}}{=} \{q \in R_p\text{: } \mathrm{Dom}^*_p(q) \cap u\} = \emptyset$,

(2) $R^+_{p,u} \stackrel{\text{def}}{=} \{q \in R^+_p\text{: } \mathrm{Dom}^*_p(q) \subseteq u\}$. $\blacksquare$

We make $GR_{p,u}$ and $R^+_{p,u}$ into partial orders by letting them inherit the order from R_p.

CLAIM 7.4: *Suppose that $\alpha \leq \omega_2$, while $p \in P'_\alpha$, $u \subseteq \mathrm{Dom}(p)$ and $r \in R^+_{p,u}$ and $s \in GR_{p,u}$. Then the following is a well defined condition in R_p: for $\beta \in \mathrm{Dom}(p)$ we let*

$$(r \cup s)(\beta) \stackrel{\text{def}}{=} \begin{cases} r(\beta) & \textit{if } \beta \in \mathrm{Dom}^*_p(r), \\ s(\beta) & \textit{if } \beta \in \mathrm{Dom}^*_p(s), \\ p(\beta) & \textit{otherwise.} \end{cases}$$

In addition, $r \cup s \geq_{\mathrm{apr}} r, s$.

Proof of the Claim: The proof is by induction on α, and the only interesting case is when $\alpha = \beta + 1$ for some even $\beta \in \mathrm{Dom}(p)$. Note that exactly one of the clauses in the definition of $(r \cup s)(\beta)$ applies. Let us work with the first one, as the other cases are similar.

Hence $\beta \in \mathrm{Dom}^*_p(r)$ and $(r \cup s) \restriction \beta \Vdash$ “$(r \cup s)(\beta) = r(\beta)$”. So we have that $(r \cup s) \restriction \beta \geq_{\mathrm{apr}} r \restriction \beta$ and $r \geq_{\mathrm{pr}} r \restriction \beta$. By Claim 4.16 $(2)^\alpha$, $r \cup s = [(r \cup s) \restriction \beta + r]$ is well defined, and the rest of the Claim is easily verified. $\blacksquare_{7.4}$

Notation 7.5: We extend our definition of “$r + s$” from 4.16 to apply also to r, s as in Claim 7.4, letting $r + s \stackrel{\text{def}}{=} r \cup s$.

Definition 7.6: Suppose that Q is a forcing notion and $M \prec (H(\chi), \in, <^*)$ is countable. We say that an increasing sequence $\bar{s} = \langle s_n\text{: } n < \omega\rangle$ of conditions in $Q \cap M$ is a **generic enough sequence for** (Q, M) iff for every formula φ with parameters in M, there are infinitely many n such that

(α) either there is no $s \geq s_n$ in Q such that $\varphi(s)$ holds, or

(β) $\varphi(s_{n+1})$.

CLAIM 7.7: *Suppose $\alpha \leq \omega_2$, while $p \in P'_\alpha$ and $u \subseteq \mathrm{Dom}(p)$.*

(1) *Suppose that $s \in GR_{p,u}$ and $r \in R^+_p$ are compatible.* Then *there are $s' \in GR_{p,u}$ and $r' \in R^+_{p,u}$ such that $s \leq s'$ and $r \leq r' + s'$.*
(Hence r is compatible with every $s'' \geq s'$ for which $s'' \in GR_{p,u}$.)

(2) *Suppose that $\{u, p, \bar{Q}, \bar{Q}', \alpha\} \subseteq M \prec (H(\chi), \in, <^*)$ is countable and $\bar{s} = \langle s_n : n < \omega\rangle$ is a generic enough sequence for $(GR_{p,u}, M)$. Further suppose $\gamma \in M \cap (\alpha + 1)$ and $r \in R^+_{p\restriction\gamma} \cap M$ is compatible with all s_n.* Then *there is $r' \in R^+_{p\restriction\gamma,u} \cap M$ such that for all large enough n we have $r \leq r' + s_n$.*

Proof of the Claim: (1) The proof is by induction on α. The interesting case is when $\alpha = \beta + 1$ for some even $\beta \in \mathrm{Dom}(p)$.

By the induction hypothesis, there are $q' \in R^+_{p\restriction\beta,u\cap\beta}$ and $t' \in GR_{p\restriction\beta,u\cap\beta}$ such that $s \restriction \beta \leq t'$ and $r \restriction \beta \leq q' + t'$.

We first work in the case that $(q' + t') \Vdash$ "$r(\beta) \geq s(\beta)$". If $\beta \notin u$ this means that $r \restriction \beta \Vdash$ "$r(\beta) = p(\beta)$". We define $r' \stackrel{\mathrm{def}}{=} q' + p$, which is well defined by Claim 4.16 $(2)^\alpha$. It is easily seen that $r' \in R^+_{p,u}$. Similarly we define $s' \stackrel{\mathrm{def}}{=} t' + s$, and check that r', s' are as required.

If $\beta \in u$, we define $r' \stackrel{\mathrm{def}}{=} q' + r$ (note that $q' \in R^+_{r\restriction\beta}$), and $s' \stackrel{\mathrm{def}}{=} t' + p$, and check that r', s' are as required.

It remains to be seen what happens in the case that it is not true that $(q' + t') \Vdash$ "$r(\beta) \geq s(\beta)$". As r and s are compatible, we can by Claim 7.1 find $z \in R_p$ such that $z \geq r, s$. By Claim 6.2, we can find $z^+ \geq^+ z$, hence $z^+ \in R^+_p$ and $z^+ \geq s$. Now we can apply the first part of the proof to z and s, and derive the desired conclusion.

If $s'' \geq s'$ and $s'' \in GR_{p,u}$, then $r \leq r + s''$ and $s'' \leq r + s''$, so r, s'' are compatible.

(2) Without loss of generality, $\alpha = \gamma$. Let

$$I \stackrel{\mathrm{def}}{=} \{s' \in GR_{p,u} : (\exists r' \in R^+_{p,u})\,(r \leq r' + s')\}.$$

Hence $I \in M$. Let n be such that when choosing s_n we have asked if there was $s' \geq s_n$ with $s' \in I$, and if possible we chose s_{n+1} to be some such s'. (In other words, either there is no $s' \geq s_n$ with $s' \in I$, or $s_{n+1} \in I$.) As r, s_n are compatible, by (1), we have chosen s_{n+1} so that for some $r' \in R^+_{p,u}$ we have $r \leq r' + s_n$. $\blacksquare_{7.7}$

Definition 7.8: Suppose that $\alpha \leq \omega_2$ and $\langle s_n : n < \omega \rangle$ and u, M are as above. Further suppose that $r \in R^+_{s_0 \restriction \alpha}$ is compatible with all s_n and $r \in M$.

(1) We define $\mathrm{Dom}(r/\bar{s}) \stackrel{\mathrm{def}}{=} \mathrm{Dom}^*_{s_0 \restriction \alpha}(r) \cap u$, and for $\beta \in \mathrm{Dom}(r/\bar{s})$

$$(r/\bar{s})(\beta) \stackrel{\mathrm{def}}{=} (t^{r(\beta)}, \bigcup_{n<\omega} \{\eta \in {}^{<\omega}2 : (r + s_n) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}\text{”}\}).$$

(2) Suppose that $n < \omega$; we define $\mathrm{Dom}(r/s_n) \stackrel{\mathrm{def}}{=} \mathrm{Dom}^*_{s_0 \restriction \alpha}(r) \cap u$ and for $\beta \in \mathrm{Dom}(r/s_n)$

$$(r/s_n)(\beta) \stackrel{\mathrm{def}}{=} (t^{r(\beta)}, \bigcup_{m<n} \{\eta \in {}^{<\omega}2 : (r + s_m) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}\text{”}\}).$$

(3) Suppose that I is a subset of P_α. We let

$$I/\bar{s} \stackrel{\mathrm{def}}{=} \{q/\bar{s} : q \in I \ \&\ q/\bar{s} \text{ defined}\}.$$

Definition 7.9: For $\alpha \leq \omega_2$, $p \in P'_\alpha$ and $q_1, q_2 \in R^+_p$ which are compatible, we define $q_1 \oplus q_2$ in R^+_p by letting for $\beta \in \mathrm{Dom}^*_p(q_1) \cup \mathrm{Dom}^*_p(q_2)$

$$(q_1 \oplus q_2)(\beta) \stackrel{\mathrm{def}}{=} (t^{q_1(\beta)} \cup t^{q_2(\beta)}, \underset{\sim}{\mathcal{T}}^{q_1(\beta)} \cup \underset{\sim}{\mathcal{T}}^{q_2(\beta)}).$$

Remark 7.10: If p, q_1, q_2 are as above, then $q_1 \oplus q_2$ is the lub of q_1, q_2 in R_p (this can be proved by induction on α).

CLAIM 7.11: *Suppose $\alpha \leq \omega_2$ and $\bar{s} = \langle s_n : n < \omega \rangle$, and u, M are as above.*

$(1)^\alpha$ *If $r \in R^+_{s_0 \restriction \alpha} \cap M$ is compatible with all s_n, then $r/\bar{s} \in P'_\alpha$, and for all large enough n we have $r/s_n \in P'_\alpha$.*

$(2)^\alpha$ *Given $q, r \in R^+_{s_0 \restriction \alpha} \cap M$ compatible with all s_n, then*

$$[q/\bar{s} \geq r/\bar{s}] \quad \text{iff } [(\forall^* n)\, (q + s_n \geq r + s_n)].$$

$(3)^\alpha$ *Suppose that $I \in M$ and $r \in R^+_{s_0 \restriction \alpha} \cap M$ is compatible with all s_n, while*

$$r \Vdash \text{“}I \text{ countable predense } \subseteq R_r\text{”}.$$

Then

$$r/\bar{s} \Vdash \text{“}I/\bar{s} \text{ countable predense } \subseteq R_{r/\bar{s}}\text{”}.$$

$(4)^\alpha$ *Suppose that $\alpha = \beta + 1$ for some $\beta \in \mathrm{Dom}(r/\bar{s})$. Further suppose that $r \in R^+_{s_0 \restriction \alpha} \cap M$ and $q \in R^+_{s_0 \restriction \beta} \cap M$ are compatible with all s_n, while $n < \omega$ and*

$$r \restriction \beta \Vdash \text{“}q \text{ determines } r(\alpha) \text{ to degree } n\text{”}.$$

Then

$$(r/\bar{s}) \restriction \beta \Vdash \text{“}q/\bar{s} \textit{ determines } (r/\bar{s}(\alpha)) \textit{ to degree } n\text{”}.$$

For r, q *as above, if* t *is such that* $q \Vdash$ “$\underset{\sim}{\mathcal{T}}^{r(\beta)} \cap {}^{<n}2 = t$”, then

$$q/\bar{s} \Vdash \text{“}\underset{\sim}{\mathcal{T}}^{(r/\bar{s})(\beta)} \cap {}^{<n}2 = t\text{”}.$$

$(5)^\alpha$ *Suppose that* $\bar{x}$ *is a canonical assignment for* s_0. *Further suppose that* $\langle p_n \colon n < \omega\rangle \in M$ *is a* $\leq_{\mathrm{pr}}$*-increasing sequence in* P'_α *with limit* p*, such that* $p \leq_{\mathrm{pr}} s_0$ *and* $\mathrm{Dom}(p) = u$. Then *for every* $n < \omega$,

$$(\forall^* l < \omega) \begin{array}{l}[\mathrm{type}_{\bar{x}}^{s_0 \restriction \alpha, n}((r + s_l) \restriction \alpha) = \\ \mathrm{type}_{\bar{x}:p_l}^{p_l \restriction \alpha, n}((r/\bar{s} + p_l) \restriction \alpha)].\end{array}$$

Proof of the Claim: We prove the claim by induction on α, proving $(1)^\alpha$–$(5)^\alpha$ simultaneously. The only eventful case of the induction is when $\alpha = \beta + 1$ for some β even.

$(1)^\alpha$ By $(1)^\beta$, we have that $(r/\bar{s}) \restriction \beta \in P'_\beta$. Without loss of generality, $\beta \in \mathrm{Dom}^*_{s_0 \restriction \alpha}(r) \cap u$. Given G which is P_β-generic and contains $(r/\bar{s}) \restriction \beta$, we have:

(a) $\underset{\sim}{\mathcal{T}}_G^{(r/\bar{s})(\beta)} \cap {}^{<\mathrm{ht}(t^{r(\beta)})}2 = t^{r(\beta)}$, as the corresponding statement about $\underset{\sim}{\mathcal{T}}^{r(\beta)}$ is forced by each $(r + s_n) \restriction \beta$.

(b) Similarly, $\underset{\sim}{\mathcal{T}}_G^{(r/\bar{s})(\beta)}$ is perfect.

(c) We show that $\underset{\sim}{\mathcal{T}}_G^{(r/\bar{s})(\beta)}$ is nowhere dense.

Given $\eta \in {}^{<\omega}2$ and n^* such that $(r + s_{n^*}) \restriction \beta \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}$”. At some stage $n \geq n^*$ we have asked if there is $s \geq s_n$ with $s \in GR_{p,u} \cap M$ such that for some $q \geq r + s_n$ with $q \in R_p^+$, and $\nu \rhd \eta$, we have $q \Vdash$ “$\nu \notin \underset{\sim}{\mathcal{T}}^{r(\beta)}$” and $q \leq q' + s$ for some $q' \in R_{p,u}^+$. By Claim 7.7(1), there was some such s which was chosen as s_{n+1}. In particular $q + s_m \geq r + s_m$ for any $m \geq n + 1$. So for no m can we have $(r + s_m) \restriction \beta \Vdash$ “$\nu \in \underset{\sim}{\mathcal{T}}^{r(\beta)}$”. Hence $\nu \notin \underset{\sim}{\mathcal{T}}_G^{(r/\bar{s})(\beta)}$.

(d) We show that $r/\bar{s}$ is simple above $(r/\bar{s}) \restriction \beta$.

Let $\bar{I} = \langle I_n \colon n < \omega\rangle$ exemplify that $r(\beta)$ is simple above $r \restriction \beta$. Without loss of generality, $I \in M$. By $(3)^\beta$+$(4)^\beta$ we have that $\langle I_n/\bar{s} : n < \omega\rangle$ exemplify that $r/\bar{s}$ is simple above $(r/\bar{s}) \restriction \beta$.

$(2)^\alpha$ Again without loss of generality we have $\beta \in \mathrm{Dom}(r/\bar{s})$. First we prove the direction from right to left.

By the induction hypothesis, $(q/\bar{s}) \restriction \beta \geq (r/\bar{s}) \restriction \beta$. By the assumption, $t^{q(\beta)} \supseteq t^{r(\beta)}$. Suppose that for some n large enough and $\eta \in {}^{<\omega}2$, we have

$$(r + s_n) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}\text{”}.$$

As $q + s_n \geq r + s_n$, we have

$$(q + s_n) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q(\beta)}\text{”}.$$

Hence $(q/\bar{s}) \restriction \beta \Vdash_{P_\beta}$ “$\underset{\sim}{\mathcal{T}}^{(q/\bar{s})(\beta)} \supseteq \underset{\sim}{\mathcal{T}}^{(r/\bar{s})(\beta)}$”.

Suppose that for some $\eta \in {}^{<\mathrm{ht}(t^{r(\beta)})}2$ and n large enough we have $(q + s_n) \restriction \beta \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{q(\beta)}$”. As $q + s_n \geq r + s_n$, we have $\eta \in t^{r(\beta)}$.

In the direction from left to right, by the induction hypothesis we have that

$$(\forall^* n)[(q + s_n) \restriction \beta \geq (r + s_n) \restriction \beta].$$

By the assumption, $t^{q(\beta)} \supseteq t^{r(\beta)}$. Suppose that for some n^* large enough, and $\eta \in {}^{<\omega}2$, we have $(q + s_{n^*}) \restriction \beta \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}$”. Let $m \stackrel{\mathrm{def}}{=} \lg(\eta)$. Let $\bar{I} = \langle I_n \colon n < \omega \rangle \in M$ exemplify that $r(\beta)$ is simple above $r \restriction \beta$. Hence, for some $z \in I_m \cap M$ which is compatible with q we have $z \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{r(\beta)}$”. Notice that such a z is compatible with every s_n. Hence $z/\bar{s}$ is defined and by $(4)^\beta$ we have $z/\bar{s} \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{(r/\bar{s})(\beta)}$”. We also have that $z/\bar{s} \geq (q \restriction \beta)/\bar{s} \geq (r \restriction \beta)/\bar{s}$, and

$$(q \restriction \beta)/\bar{s} \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{r/\bar{s}(\beta)} \Longrightarrow \eta \in \underset{\sim}{\mathcal{T}}^{q/\bar{s}(\beta)}\text{”}.$$

So $z/\bar{s} \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{q/\bar{s}(\beta)}$”. As $z/\bar{s}$ and $q + s_n$ are compatible for all large enough n, it must be that for some n^* large enough

$$(q + s_{n^*}) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q(\beta)}\text{”}$$

(by the genericity of $\bar{s}$).

Now suppose that for some $\eta \in {}^{<\mathrm{ht}(t^{r(\beta)})}2$ and n large enough we have $(q+s_n) \restriction \beta \Vdash$ “$\eta \in \underset{\sim}{\mathcal{T}}^{q(\beta)}$”. Hence $\eta \in \underset{\sim}{\mathcal{T}}^{q/\bar{s}(\beta)}$, so, as $q/\bar{s} \geq r/\bar{s}$, it must be that $\eta \in t^{r(\beta)}$.

$(3)^\alpha$ Certainly $I/\bar{s}$ is countable, and by $(2)^\alpha$ we also know that $r/\bar{s} \Vdash$ “$I/\bar{s} \subseteq R_{r/\bar{s}}$”.

We show that

$$I/\bar{s} \text{ is a predense } \subseteq R_{r/\bar{s}}.$$

Let $I = \{q_l \colon l < \omega\}$. Suppose that $z^* \in R_{r/\bar{s}}$, and we wish to show that z^* is compatible with some $q_l/\bar{s}$. Without loss of generality, $z^* \in R^+_{r/\bar{s}}$. If $q_l/\bar{s}$ is defined and $z^* \restriction \beta$ is compatible with $(q_l/\bar{s}) \restriction \beta$, the only way that z^* and $q_l/\bar{s}$ can turn out to be incompatible is that one of the following happens:

(i) Neither is $t^{z(\beta)}$ an end extension of $t^{(q/\bar{s})(\beta)}$, nor the other way around,

(ii) or $t^{z(\beta)}$ is an end extension of $t^{(q/\bar{s})(\beta)}$, but for some $\eta \in {}^{<\mathrm{ht}(t^{z(\beta)})}2$ which is not in $t^{z(\beta)}$, we have

$$(z \restriction \beta) \oplus (q_l/\bar{s}) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q_l/\bar{s}(\beta)} \cup \underset{\sim}{\mathcal{T}}^{z(\beta)}\text{”},$$

(iii) or $t^{q/\bar{s}(\beta)}$ is an end extension of $t^{z(\beta)}$, but for some $\eta \in {}^{<\mathrm{ht}(t^{q(\beta)})}2$ which is not in $t^{q(\beta)}$ we have

$$(z \restriction \beta) \oplus (q_l/\bar{s}) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q_l/\bar{s}(\beta)} \cup \underset{\sim}{\mathcal{T}}^{z(\beta)}\text{”}.$$

When choosing s_n, for some large enough n we have asked if there is $z \in P'_\alpha$ with $\mathrm{Dom}(z) = \mathrm{Dom}^*_{s_0 \restriction \alpha}(r) \cap u$ and such that

(A) $z \restriction \beta \geq (r/s_n) \restriction \beta$.

(B) $(\forall \gamma \in \mathrm{Dom}(z))(\exists t_\gamma)\,(\Vdash_{P_\gamma} \text{“}\underset{\sim}{t}^{z(\gamma)} = t_\gamma\text{”})$.

(C) For all $l \leq n$ one of the following happens:

  (a) $z \restriction \beta$ is incompatible with $(q_l/s_n) \restriction \beta$;

  (b) neither is $t^{z(\beta)}$ an end extension of $t^{q_l(\beta)}$, nor is $t^{q_l(\beta)}$ an end extension of $t^{z(\beta)}$;

  (c) $t^{z(\beta)}$ is an end extension of $t^{q_l(\beta)}$, but for some $\eta \in {}^{<\mathrm{ht}(t^{z(\beta)})}2$ which is not in $t^{z(\beta)}$, we have

$$(z \restriction \beta) \oplus (q_l/s_n) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q_l/s_n(\beta)} \cup \underset{\sim}{\mathcal{T}}^{z(\beta)}\text{”};$$

  (d) $t^{q_l(\beta)}$ is an end extension of $t^{z(\beta)}$, but for some $\eta \in {}^{<\mathrm{ht}(t^{q(\beta)})}2$ which is not in $t^{q_l(\beta)}$ we have

$$(z \restriction \beta) \oplus (q_l/s_n) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q_l/s_n(\beta)} \cup \underset{\sim}{\mathcal{T}}^{z(\beta)}\text{”}.$$

If after some n^* the answer to the above question was never positive, this means that z^* could not have been used as a witness, which means that z^* is compatible with some $q_l/\bar{s}$.

Suppose that the answer was positive at some large enough n, and let this be exemplified by some z. Without loss of generality we have $z \in M$. We can find $m > n$ such that, with q_m in place of q_l above, neither of the first two possibilities happen. So suppose the third one does. Hence for some $k < m$ we have that

$$z \restriction \beta + (q_m + s_k) \restriction \beta \Vdash \text{“}\eta \in \underset{\sim}{\mathcal{T}}^{q_m(\beta)} \cup \underset{\sim}{\mathcal{T}}^{z(\beta)}\text{”}$$

for some $\eta \in {}^{<\mathrm{ht}(t^{z(\beta)})}2 \setminus t^{z(\beta)}$. But this is a contradiction with $z \restriction \beta$ being compatible with $q_m + s_k$.

$(4)^\alpha$ Similar.

$(5)^\alpha$ For $l < \omega$ let

$$\Upsilon^*_l \stackrel{\mathrm{def}}{=} \mathrm{type}_{\bar{x}}^{s_0 \restriction \alpha, n}((r + s_l) \restriction \alpha) \quad \text{and}$$
$$\Upsilon'_l \stackrel{\mathrm{def}}{=} \mathrm{type}_{\bar{x}:p_l \restriction \alpha}^{p_l \restriction \alpha, n}((r/\bar{s} + p_l) \restriction \alpha).$$

We show that for large enough l we have $\Upsilon^*_l = \Upsilon'_l$, by comparing the corresponding 9 coordinates. It is easy to see that for any l we have that

$$n^{[\Upsilon^*_l]} = n^{[\Upsilon'_l]} = n, \quad o^{[\Upsilon^*_l]} = o^{[\Upsilon'_l]} = 1, \quad \beta^{[\Upsilon^*_l]} = \beta^{[\Upsilon'_l]} = \beta,$$
$$t^{[\Upsilon^*_l]} = t^{[\Upsilon'_l]} = t^{r(\beta)} \quad \text{and}$$
$$\bar{\varepsilon}^{[\Upsilon^*_l]} = \bar{\varepsilon}^{[\Upsilon'_l]} = \mathrm{Dom}(r/\bar{s}) \cap \alpha \text{ in the increasing enumeration.}$$

We now prove that for large enough l we have $w^{[\Upsilon^*_l]} = w^{[\Upsilon'_l]}$. Assume $m < n$.

When choosing s_l's, we have infinitely often asked if there is $s' \geq_{\mathrm{pr}} s_l$ and $q \geq r$ such that

(i) $s' \in GR_{s_0,u}$,

(ii) $(q + s') \restriction \beta \in {}^{\bar{x}}A^\beta_m$,

(iii) for some $l' \geq l$ we have $(q/s') \restriction \beta \in {}^{\bar{x}:p_l \restriction \beta}A^\beta_m$,

and, if possible, we have chosen some such s' as s_{l+1}.

POSSIBILITY 1: For some l large enough we chose s_{l+1} to satisfy (i)–(iii) above with s_{l+1} in place of s'.
Hence there is q which witnesses the choice. By $(2)^\alpha$, we have $q/\bar{s} \geq r/\bar{s}$, so $m \in w^{[\Upsilon^*_l]} \cap w^{[\Upsilon'_l]}$.

POSSIBILITY 2: For no large enough l could we have chosen s_{l+1} to satisfy (i)–(iii) above with s_{l+1} in place of s'.

Suppose that l is large enough and $m \in w^{[\Upsilon_l^*]}$, as exemplified by q. Without loss of generality $q \in M$. Hence $q \geq_{\rm apr} (r + s_l) \restriction \beta$ and $q \in R^+_{s_0 \restriction \beta}$. We have $(q/\bar{s}) \restriction \beta \geq (r/\bar{s}) \restriction \beta$. Let q, i be such that ${}^{\bar{x}}A^{\beta}_m = z/E^{s_0 \restriction \beta, i}_{\bar{x}}$. Without loss of generality we have $z \in M$. Hence, by the induction hypothesis we have

$$^{\bar{x}:p_l \restriction \beta}A^{\beta}_m = (z/\bar{s} + p_l) \restriction \beta / E^{p_l \restriction \beta, i}_{\bar{x}:p_l}.$$

By the induction hypothesis, for large enough l we have $q/\bar{s} \in {}^{\bar{x}:p_l}A^{\beta}_m$. This is a contradiction. Hence $m \notin w^{[\Upsilon_l^*]}$ for all large enough l.
We similarly show that $m \notin w^{[\Upsilon_l']}$ for all large enough l.
Other parts of the claim are checked similarly. $\blacksquare_{7.11}$

8. Obtaining $\clubsuit$ in V^P

CLAIM 8.1: $V^P \models \clubsuit$.

Proof of the Claim:

Definition 8.2: Suppose that $\bar{N}$ is a sequence of elementary submodels of $\langle H(\chi), \in, <^*_\chi \rangle$ and $\bar{a}$ a finite sequence in $\bar{N}(0)$. We say that an $x \in H(\chi)$ is **chosen canonically for** $(\bar{N}, \bar{a})$, if the choice of x depends only on the isomorphism type of $(\bar{N}, \bar{a})$ as a submodel of $(H(\chi), \in, <^*_\chi, \vec{a})$, where $\vec{a}$ is a finite list of constant symbols (interpreted in $\bar{N}(0)$ as $\bar{a}$).

MAIN CLAIM 8.3: (1) *Given a sequence $\bar{N} = \langle N_n : n < \omega \rangle$ of countable elementary submodels of $\langle H(\chi), \in, <^*_\chi \rangle$ with $N_n \in N_{n+1}$ for all n, and $\bar{Q}, \bar{Q}', \underset{\sim}{\tau} \in N_0$ and $p \in N_0 \cap P$ such that*

$$p \Vdash \text{“} \underset{\sim}{\tau} \in [\omega_1]^{\aleph_1} \text{”},$$

let $\bar{a} = \langle \underset{\sim}{\tau}, p, \bar{Q}, \bar{Q}' \rangle$ and let $\delta \stackrel{\rm def}{=} \bigcup_{n<\omega} (N_n \cap \omega_1)$.

Then *there is*

(a) *a strictly increasing sequence $\bar{\beta} = \bar{\beta}(\bar{N}, \bar{a}) = \langle \beta_n : n < \omega \rangle$ with $\sup_{n<\omega} \beta_n = \delta$, which is chosen canonically for $(\bar{N}, \bar{a})$, and*

(b) *a condition $r^{\oplus} = r^{\oplus}_{\bar{N}, \bar{a}} \geq p$, with $r^{\oplus} \Vdash$ “$\{\beta_n : n < \omega\} \subseteq \underset{\sim}{\tau}$”.*

(2) *Values of $\beta_n \stackrel{\rm def}{=} \bar{\beta}(\bar{N}, \bar{a})(n)$ for $n < \omega$, and the fact that there is an $r^{\oplus} \geq p$ such that $r^{\oplus} \Vdash$ “$\{\beta_n : n < \omega\} \subseteq \underset{\sim}{\tau}$” only depend on the isomorphism type of $(\bar{N}, \bar{a})$ as a submodel of $(H(\chi), \in, <^*_\chi, \vec{a})$.*

Proof of the Main Claim: (1) Let $N_\omega \stackrel{\rm def}{=} \bigcup_{n<\omega} N_n$.

SUBCLAIM 8.4: *Suppose that $\bar{N}$ and $\bar{a}$ are as in the statement of the Main Claim 8.3 and N_ω as defined above. Let $\xi_{\bar{N}} \stackrel{\text{def}}{=} \text{otp}(N_\omega \cap \omega_2 \cap ODD)$ and let h be the order isomorphism exemplifying this. Let $\langle u^*_n: n < \omega\rangle$ be the $<^*_\chi$-first increasing sequence of finite sets such that $u_n \stackrel{\text{def}}{=} h^{-1}(u^*_n) \subseteq N_n$ and $\bigcup_{n<\omega} u^*_n = \xi_{\bar{N}}$. Let $\{\varphi_n: n < \omega\}$ be the $<^*_\chi$-first enumeration of the first order formulas with parameters in N_ω, each formula appearing infinitely often, and such that the parameters of φ_n are contained in N_n.*

Then *there are sequences*

$$\bar{p} = \bar{p}_{\bar{N},\bar{a}} = \langle p_n: n < \omega\rangle \quad \text{and} \quad \bar{q} = \bar{q}_{\bar{N},\bar{a}} = \langle q_n : n < \omega\rangle$$

chosen canonically for $\bar{N}$ and $\bar{a}$ such that:

(i) $q_0 = p_0 = p$.

(ii) $p_{n+1} \geq_{\text{pr}} p_n$.

(iii) $p_n \leq_{\text{apr}} q_n$.

(iv) $p_n, q_n \in N_{n+1}$.

(v) *For all n and $\alpha \in u_n$, we have that*

$$p_{n+1} \restriction \alpha \Vdash \text{“} \underset{\sim}{f}^{\,p_{n+1}(\alpha)}(x) < \underset{\sim}{f}^{\,p_n(\alpha)}(x \restriction (\delta^*(p_n) + 1) + 1/2^n\text{”},$$

for all $x \in \underset{\sim}{w}_{\underset{\sim}{g}_{\alpha-1}(n)}^{\underset{\sim}{T}^{\alpha}_{\delta^*(p_{n+1})}}$.

(vi) *For every n*

either

(α) *There is no $p' \geq_{\text{pr}} p_n$ and $q' \geq_{\text{apr}} p'$ such that $\varphi_n(p', q)$ and* (v) *above holds with p' in place of p_{n+1},*

or

(β) (p'_n, q_n) *are the $<^*_\chi$-first elements of $H(\chi)$ which exemplify that* (α) *does not happen, with p'_n in place of p' and q_n in place of q.*

Proof of the Subclaim: The proof is straightforward. Construct p_n, q_n by induction on n, the step at the stage $n = 0$ being given. At the stage $n+1$, we are given p_n and we consider φ_n. If option (α) holds, just let $p_{n+1} = q_{n+1} \stackrel{\text{def}}{=} p_n$. If ($\beta$) holds, then find (p_{n+1}, q_{n+1}) as described in (β), and note that $p_{n+1}, q_{n+1} \in N_{n+2}$. $\blacksquare_{8.4}$

SUBCLAIM 8.5: *Suppose that $\bar{N}$, $\bar{a}$ and $\bar{p} = \bar{p}_{\bar{N},\bar{a}}$ are as in the Claim 8.4.*

Then *there is a canonically chosen condition $p_\omega = p_{\bar{N},\bar{a}}$ such that for all n we have $p_n \leq_{\text{pr}} p_\omega$, while* $\text{Dom}(p_\omega) = N_\omega \cap \omega_2$ *and* $\delta^*(p_\omega) = N_\omega \cap \omega_1$.

Proof of the Subclaim: Use the same argument as the one used in Claim 5.1 to prove $(2)^\alpha$ at the stages α of countable cofinality. $\blacksquare_{8.5}$

There is $r \geq p_\omega$ such that $r \Vdash$ "$\beta \in \underset{\sim}{\tau}$" for some $\beta > \delta$. By Observation 4.13 and Claim 6.2 there are s_0, r^* such that

(i) $p_\omega \leq_{\rm pr} s_0 \leq^+_{\rm apr} r^*$, and

(ii) $[\alpha \in \mathrm{Dom}(r^*)\ \&\ \neg(r^* \restriction \alpha \text{“} \Vdash r^*(\alpha) = s_0(\alpha)\text{”})] \Longrightarrow \alpha \in \mathrm{Dom}(p_\omega)$.

(iii) For some $\beta^* > \delta$ we have $r^* \Vdash$ "$\beta^* \in \underset{\sim}{\tau}$".

Now let M be countable $\prec \langle H(\chi), \in, <^*_\chi\rangle$ such that $\{\bar{N}, s_0, r^*, \beta^*\} \subseteq M$.

Let $v \stackrel{\rm def}{=} \{\alpha\colon \neg(r^* \restriction \alpha \Vdash \text{“}r^*(\alpha) = s_0(\alpha)\text{”})\}$, hence v is finite $\subseteq \mathrm{Dom}(p_\omega)$.

Let $\bar{s} = \langle s_n\colon n < \omega\rangle$ be a generic enough sequence for $(\mathrm{GR}_{s_0, \mathrm{Dom}(p_\omega)}, M)$. Let $\bar{x}$ be a canonical assignment for s_0.

Definition of $\bar{\beta}$ and $r^\oplus$: By induction on $n < \omega$ we shall define β_n, as well as natural numbers m_n and conditions r_n.

"$n = 0$." We let $m_0 = n_0$ and $\beta_0 \stackrel{\rm def}{=} N_{m_0} \cap \omega_1$.

"$n + 1$." Given are m_n and β_n.

Let $m' \stackrel{\rm def}{=} m'_{n+1}$ be the first large enough integer $> m_n$ so that

$$\mathrm{type}^{s_0, n}_{\bar{x}}(r^* + s_{m'}) =$$
$$\mathrm{type}^{p_{m'}, n}_{\bar{x}:p_{m'}}(r^*/\bar{s} + p_{m'}) \stackrel{\rm def}{=} \Upsilon_n.$$

We now consider the formula $\psi^n(x_0, x_1)$ saying that

(I) $p_{m'} \leq_{\rm pr} x_0 \leq_{\rm apr} x_1$ and

(II) $x_1 \Vdash$ "$\gamma \in \underset{\sim}{\tau}$" for some $\gamma > N_{m'} \cap \omega_1$ and

(III) we have $\mathrm{type}^{x_0, n}_{\bar{x}:p_{m'}}(x_1) = \Upsilon_n$.

Let m_{n+1} be the first $m > m'$ such that $\varphi_m = \psi_n$. Hence we have chosen $(p_{m_{n+1}+1}, q_{m_{n+1}+1})$ so that $\psi_n((p_{m_{n+1}+1}, q_{m_{n+1}+1}))$ holds, as is exemplified by $(s_0, r^* + s_m)$.

Let $r_n \stackrel{\rm def}{=} q_{m_n} + p_\omega$, for $n < \omega$. We shall define $r^\oplus$ so that $r^\oplus \geq r_n$ for all n. Hence $r^\oplus \Vdash$ "$\{\beta_n : n < \omega\} \subseteq \underset{\sim}{\tau}$".

The Main Point: Why does such $r^\oplus$ exist? All r_n are elements of $R^+_{p_\omega}$ and, by the definition of Υ_n, each has the property that $r_n\, E^{p_\omega, n}_{\bar{x}:p_\omega}(r^*/\bar{s} + p_\omega)$. By Lemma 2.13, there must be $\check{r} \in R_{p_\omega}$ which is a common upper bound to $\{r_n : n < \omega\}$. Let $r^\oplus \stackrel{\rm def}{=} \check{r} + p_\omega$.

Proof of the Main Claim continued: (2) It suffices to observe the following

Observation 8.6: Given $\bar{N}, \bar{a}$ as in Main Claim 8.3, let $\bar{p}$ and $\bar{q}$ be as in Subclaim 8.4. Let $\bar{z}$ be a canonical assignment for p_ω. Suppose that $X \in [\omega]^{\aleph_0}$ is such that

$\{q_n: n \in X\}$ has an upper bound in $\mathfrak{B}_{p_\omega, \bar{z}}$. Suppose that $f: (\bar{N}, \bar{a}) \to (\bar{N}', \bar{a}')$ is an isomorphism.

Then

$$\{f(q_n) : n \in X\} \text{ has an upper bound in } \mathfrak{B}_{\bigcup_{n<\omega} f(p_n), \bigcup_{n<\omega} f(\bar{z}: p_n)}. \quad \blacksquare_{8.3}$$

Now we can finish proving Claim 8.1 and so Theorem 3.1. Let

$$\mathfrak{A} \stackrel{\text{def}}{=} (H(\chi), \in, <^*_\chi, p, \underset{\sim}{\tau}, \bar{Q}, \bar{Q}'),$$

where $p, \underset{\sim}{\tau}, \bar{Q}, \bar{Q}'$ are constant symbols. We arrange $\diamondsuit$ in V in this form:

There is a sequence

$$\langle \bar{N}^\delta = \langle N^\delta_i : i < \delta \rangle : \delta < \omega_1 \text{ limit} \rangle$$

such that

1. N^δ_i is a countable elementary submodel of $\mathfrak{A}$, with $N^\delta_i \cap \omega_1 < \delta$, $\bar{N}^\delta \restriction i \in N^\delta_{i+1}$.
2. $\bar{N}^\delta$ is continuously increasing.
3. For every continuously increasing sequence $\langle N_i: i < \omega_1 \rangle$ of countable elementary submodels of $\mathfrak{A}$, there is a stationary set of δ such that the isomorphism type of $\langle N_i: i < \delta \rangle$ is the same as that of $\langle N^\delta_i: i < \delta \rangle$.

For $\delta < \omega_1$ a limit ordinal, we choose the $<^*_\chi$-first increasing ω-sequence $\langle \epsilon^\delta_n: n < \omega \rangle$ of ordinals such that $\sup_{n<\omega} \epsilon^\delta_n = \delta$ and $\epsilon^\delta_0 = 0$. We define sets A_δ for such δ as follows. Let $N^\delta \stackrel{\text{def}}{=} \bigcup_{i<\delta} N^\delta_i$.

If $N^\delta \cap \omega_1 = \delta$, $p^{N^\delta_0} \in P' \cap N^\delta_0$ and $p^{N^\delta_0} \Vdash$ "$\underset{\sim}{\tau}^{N^\delta_0} \in [\omega_1]^{\aleph_1}$", then

$$A_\delta \stackrel{\text{def}}{=} \operatorname{Rang}(\bar{\beta}(\langle N^\delta_{\epsilon^\delta_n}: n < \omega \rangle)).$$

Otherwise, we let A_δ be the range of any cofinal ω-sequence in δ.

We claim that $\langle A_\delta: \delta \text{ limit } < \omega_1 \rangle$ is a ♣-sequence in V^P.

So suppose that $p^* \Vdash$ "$\underset{\sim}{\tau}^* \in [\omega_1]^{\aleph_1}$" and $p \in P'$. We fix a continuously increasing sequence $\bar{N} = \langle N_i: i < \omega_1 \rangle$ of countable elementary submodels of $\mathfrak{A}$ such that $\bar{Q} = \bar{Q}^{N_0}, p^{N_0} = p^*, \underset{\sim}{\tau}^{N_0} = \underset{\sim}{\tau}, \bar{Q}' = [\bar{Q}']^{N_0}$ and $\bar{N} \restriction i \in N_{i+1}$ for all $i < \omega_1$. Then

$$C \stackrel{\text{def}}{=} \{\delta < \omega_1: \delta \text{ limit and } N_\delta \cap \omega_1 = \delta\}$$

is a club of ω_1. Hence there is $\delta < \omega_1$ such that $\langle N_i: i < \delta \rangle$ and $\langle N^\delta_i: i < \delta \rangle$ have the same isomorphism type. So A_δ is defined by the first clause in its definition.

Hence, by Main Claim 8.3(2), we have $A_\delta = \mathrm{Rang}(\bar{\beta}(\langle N_{\epsilon^\delta_n} : n < \omega \rangle))$, while $\bar{r}(\bar{N} \restriction \delta)$ has an upper bound, say $r^\otimes$. Now $r^\otimes \geq p$ and $r^\otimes \Vdash$ "$A_\delta \subseteq \underset{\sim}{\mathcal{T}}^*$".
$\blacksquare_{8.1}$ $\blacksquare_{3.1}$

Remark 8.7: Note that the club sequence $\langle A_\delta : \delta < \omega_1 \rangle$ we obtained for the final model is in fact a sequence in V.

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